Geometric Modeling

Lecture #1: September 9, 2002
Lecturer: Denis Zorin
Scribe: Elif Tosun

In this lecture we review some basic facts from classical differential geometry of curves, and start considering surfaces; we discuss surfaces in more detail in a future lecture.

1 Curves

A common representation of a curve is the parametric representation. Parametric representation is writing the curve as a function of one parameter such that $y(t) : \mathbf{R} \rightarrow \mathbf{R}^n$, for $n = 2, 3, \ldots$. However, not every function of this type describe what we intuitively consider a curve. Example: if the function is constant, it defines a single point.

Intuitively, a function $y(t)$ describes a curve if

1. the function is continuous,
2. the image of function y is one-dimensional.

A **one dimensional manifold** can be defined as follows: for each point x in the domain, there exists an interval U_x around x such that $y|U_x$ is one-to-one and continuous.

There are some "strange" curves that formally satisfy the definition above but intuitively are not really curves, for example, the Hilbert curve and the Snowflake curve. The Hilbert curve traverses all points in a square. See Figure 1 for the first two steps of the recursive construction of the Hilbert curve, and the curve after several steps.

1.1 C^1-Curves

Now, we restrict our concept of curves to "smooth curves" to avoid such strange curves mentioned above. The intuitive definition of a smooth curve is based on the tangents of a curve. A curve is smooth if you move any two points along a curve towards a common limit position, and the line connecting these two points converges to a limit direction (tangent), which is the same no matter how we move the points.

A more convenient definition is based on the parametric representation $y(t) : \mathbf{R} \rightarrow \mathbf{R}^n$. A curve is smooth if

1. The derivative of the function defining the curve exists,
2. The derivative is not equal to zero $\frac{dy}{dt} \neq 0$.

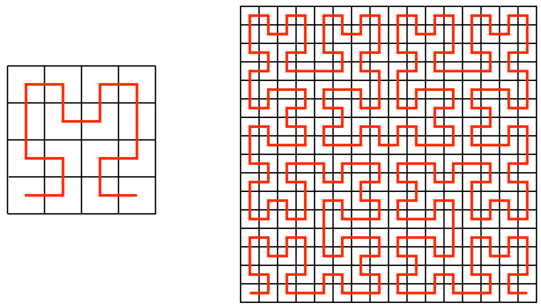

Figure 1: The first step of the recursive construction of a Hilbert Curve.

Although these properties are sufficient, they are not necessary for smoothness. A more accurate definition, also referred to as C^1**-continuity**, is the following:

There exists a choice of parameterization such that the derivative exists and is not equal to zero for any point of the domain.

Unit tangent is a tangent of length 1, namely $\frac{dy}{dt}/|\frac{dy}{dt}|$.

Example 1. This example shows that if the derivative vanishes, a differentiable function can produce an non-smooth curve: $y(t) = [t^2, t^3]$. When plotted, this curve has a corner at the point where the derivative vanishes ($t = 0$). See Figure 2.

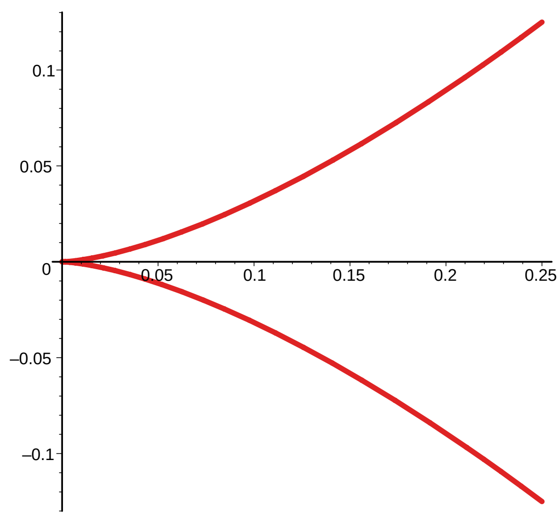

Figure 2: $y(t) = [t^2, t^3]$.

Example 2. On the other hand, if the derivative vanishes, the curve still can be smooth: $y(t) = [t^3, t^3]$. In this case, we can change the parameterization so that the derivative does not vanish: for $s = t^3$ the curve becomes $y(s) = [s, s]$; the derivative $\frac{dy}{ds} = 1$, for all values of s, so the curve is C^1-continuous.

Example 3. While C^1-continuous curves cannot be as strange as the Hilbert curve, they still may have unpleasant properties. Consider for example $y(t) = [t, t^2 \sin(\frac{1}{t})]$. This is an interesting example of a curve that intersects a line infinitely many times, see Figure 3. As t approaches zero, the x-axis is crossed infinitely many times. But because the curve is sandwiched between the two parabolas with common tangent at zero, the tangent to the curve still exists. Infinite number of intersections is bad news computationally, because any curve-curve intersection algorithm would fail to find all of them in finite time.

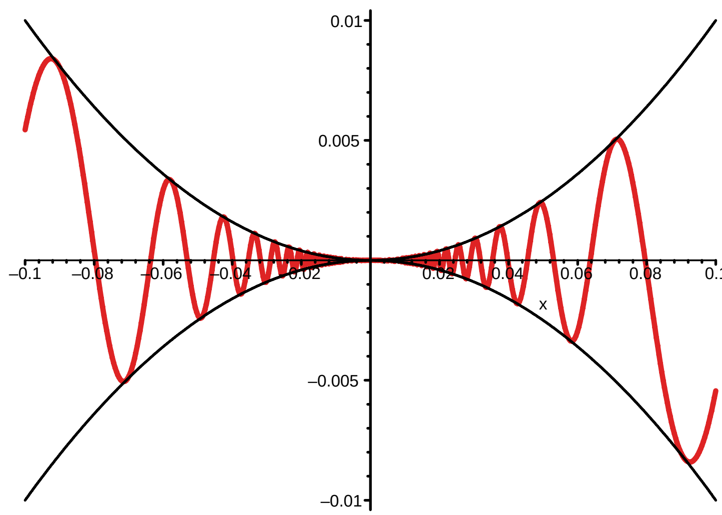

Figure 3: $y(t) = [t, t^2 \sin(\frac{1}{t})]$.

1.2 C^2-Curves.

These are curves where the first and the second derivative exists. Therefore, a curve is C^2-**continuous** if C^1 and y'' exists for some choice of parameterization.

Geometrically, if we pick three points, they uniquely define a circle. We saw earlier that if we bring two points together we get a limit tangent. If we bring three points A, B and C together, for a C^2 curve we get a limit **osculating** circle with radius R, which is the limit radius of a circle passing through A, B, and C as they are pulled together. R can be infinite. **Curvature** is $\frac{1}{R}$ by definition.

Clearly, if the three points are on a straight line then $R = \infty$, so $\frac{1}{R} \to 0$.

Curvature also can be defined defined as the speed of unit tangent rotation.

Unit speed parameterization is a choice of parameterization $y(s)$ such that $|dy/ds| = 1$. In this parameterization curvature is $|d^2y/ds^2|$.

2 Surfaces

We define a surface, based on the following representation; $y(u,v)\ :\ \mathbf{R}^2 \rightarrow \mathbf{R}^3$. As it is the case for the curves, the geometric shape if the image of $y(u,v)$ may be not intuitively a surface: it can be a point or a curve.

We want to know how we can distinguish between a curve and a surface given a parameterization. For example given $y = [f_1(u,v), f_2(u,v), f_3(u,v)]$. If $f_1 = f_2 = f_3$ then it is a straight line. Surfaces are distinguished by the following property: for each point x_0 in the domain, there exists a disk $B^{\epsilon}_{x_0}$ around x_0 such that $y|B^{\epsilon}_{x_0}$ is one-to-one and continuous.

For differentiable parameterizations $y(u,v)$ the property follows from the following sufficient condition for surface C^1-continuity:

if $y(u,v)$ is continuously differentiable then $\frac{\partial y}{\partial u}$ x $\frac{\partial y}{\partial v} \neq 0$.

This is also equivalent to saying that the rank of matrix

$$\begin{pmatrix} \frac{\partial y_1}{\partial u} & \frac{\partial y_1}{\partial v} \\ \frac{\partial y_2}{\partial u} & \frac{\partial y_2}{\partial v} \\ \frac{\partial y_3}{\partial u} & \frac{\partial y_3}{\partial v} \end{pmatrix}$$

is equal to 2.