Fields on Symmetric Surfaces

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Appendix: Proofs of statements

Proof of Proposition 2. First, consider a stationary point p of g. As shown [Montgomery and Zippin 1955], there is a neighborhood U of p and a choice of smooth coordinates $h : U \to \mathbb{R}^2$ system on U such that g in these coordinates is a linear transformation A_g^p, i.e. $g = h^{-1} \circ A_g \circ h$. It follows that $Dg(p)$ has the form $V(p)A_g^p V(p)^{-1}$ where $V(p)$ is the differential of the transformation h at point p. As $Dg(p)^2 = I$ at a stationary point, it follows that $(A_g^p)^2 = I$. All such matrices have two eigenvalues, and both its eigenvalues satisfy $\lambda^2 = 1$.

*O*rientation-preserving g. In this case, we show that g cannot be a reflection. In this case, both eigenvalues are either 1 or -1. Consider the set $M^1(g)$ of all stationary points p with both eigenvalues of A_g^p equal to 1, and let $M^2(g)$ be the set of all stationary points with both eigenvalues equal to -1. For points from $M^1(g)$, $A_g^p = I$, and $g = h^{-1} \circ h$ is identity on U, i.e. any stationary point of this type has an open neighborhood of stationary points of the same type. We conclude that $M^1(g)$ is open. At any stationary point p from $M^2(g)$, A_g^p is $-I$, i.e. g has a single stationary point in U (p itself): $M^2(g)$ consists of isolated points. On the other hand, the set of all stationary points $M(g) = M^1(g) \cup M^2(g)$ is closed, as the limit of any sequence of stationary points is stationary by continuity of g. The limit of a sequence of points from $M^1(g)$ has to be a point from $M^1(g)$, as all points in $M^2(g)$ are isolated, so the limit of points in $M^1(g)$ is also in $M^1(g)$. We conclude that $M^1(g)$ is both open and closed. As we consider connected surfaces, an open/closed subset of an open surface has to be either empty or the whole surface. In the former case, $M(g) = M^2(g)$, i.e. the stationary set consists of isolated points. A set of isolated points cannot separate the nonstatationary subset into two disconnected components, so we conclude that this case is not possible for generalized reflections. In the latter case ($M(g) = M^1(g)$ is the whole surface), the map g is an identity, i.e. this case is not possible for reflections either.

*O*rientation-reversing g. If g is orientation-reversing, at every stationary point, its differential Dg and linear form A has eigenvalues 1 and -1, and in $h(U)$ the stationary set of A is a line ℓ, corresponding to the stationary curve $h^{-1}(\ell)$ of g. As this holds for any stationary point, the stationary curve can be extended indefinitely to an embedding of the real line or a circle in M, forming a connected component of the stationary set. As the stationary set is closed, its connected components are also closed. But an embedding of a real line in a compact manifold cannot be closed; we conclude that the stationary set consists of embeddings of circles.

Consider a point p in one of the connected components M_1 of the non-stationary set M' of M, mapped to a component M_2. Consider the set of all points in M_1 mapped to M_2, i.e. $M_1 \cap g^{-1}(M_2)$. As M_2 is both open and closed in M', so is $g^{-1}(M_2)$ by continuity of g. Thus, $M_1 \cap g^{-1}(M_2)$ is also open and closed, so it has to coincide with all of M_1 as M_1 is connected, i.e. $g(M_1) \in M_2$. As $g(g(p))$ is p, by a similar argument, $g(M_2) \in M_1$, so M_2 and M_1 are mapped to each other, and $g(M_1) = M_2$. Consider a point p on the boundary of M_1. As locally g acts as a linear reflection, mapping one part of the neighborhood U of p to the other, U has to consist of two disconnected parts from M_1 and M_2, i.e., any point on the boundary of M_1 separates it from M_2. Then the union of M_1, M_2 and their boundary is closed in M and has no boundary, i.e., it has to coincide with M.

Proof of Lemma 1. By Proposition 2, the differential Dg_p at a stationary point p has two eigenvalues -1 and 1 (see proof above). Let e_1 be the eigenvector corresponding to eigenvalue 1: e_1 is a stationary direction of Dg_p. Now let us assume a change of coordinate system on T_p that aligns the first coordinate axis to e_1. If we express Dg_p with respect to the new frame, it must necessarily have the form:

$$\begin{bmatrix} 1 & c \\ 0 & d \end{bmatrix}.$$

Since $\det Dg_p = -1$ we necessarily have $d = -1$.

Proof of Corollary 3. Let $g : M \to M$ be a diffeomorphism such that $g^2 = Id$, M has sphere topology. As the stationary set partitions M into two connected domains, each has to be a disk, and so the curve is a topological circle (as it bounds a disk). Let $b : M \to S$ be a one-to-one mapping from the surface to a sphere. Let $\phi : S \to S$ be a homeomorphism of the sphere to itself that maps the stationary set of $b \circ g \circ b^{-1}$ to a great circle. It follows that $\phi \circ b \circ g \circ b^{-}1 \circ \phi^{-1}$ has the circle as the stationary line. There is a stereographic projection P from the sphere to the plane mapping this circle to a line, say the x axis. Let $h = P \circ \phi \circ b \circ g \circ b^{-1} \circ \phi^{-1} \circ P^{-1}$, this is a homeomorphism from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$ such that the x-axis is stationary, and it swaps two halves of the plane. Clearly, $h^2 = Id$. Let R be the reflection of the plane that maps y to $-y$. Then $R \circ h$ is a homeomorphism that maps each half-plane to itself. Let H_1 and H_2 be the two half-planes. Define the coordinate change f on the plane as Id on H_1, and $R \circ h$ on H_2. Then for x in H_1, $h(x) = h \circ R \circ R \circ Id = h^{-1} \circ R^{-1} \circ R \circ Id = (Rh)^{-1} \circ R \circ Id = f^{-1} \circ R \circ f$, and for $x \in H_2$, again, $h(x) = Id \circ R \circ R \circ h = f^{-1} \circ R \circ f$, in other words, we got the factorization we wanted.

Proof of Lemma 4. Using the expression for R^g, we observe that it defines an analytic dependence of R^g on Dg, unless $det(Dg + Dg^T - Tr(Dg)I) = 0$, which, as can be seen by direct calculation, only happens if Dg is a similarity transformation. However, as Dg is orientation-reversing, this is not possible. Since $g^2 = Id$ then $Dg_{g(p)} Dg_p = I$. Since at a point p, $Dg_p = R^g S^g$, then $D_{g_{g(p)}} = Dg_p^{-1} = S^{g^{-1}} (R^g)^T = (R^g)^T S'$ with $S' = R^g S^{g^{-1}} (R^g)^T$ symmetric positive definite, so the closest orthogonal transform to $Dg_{g(p)}$ is $R^g(p)^T$, which implies the second statement of the lemma.

Proof of Proposition 6. Let us assume v is not singular at p, and let w be one of the N vectors of $v(p)$. Since v is stationary (as a N-symmetry field) for R^g, then $R^g w$ must also be one of the vectors of $v(p)$, i.e., w and $R^g w$ must form an angle of $2k\pi/N$ for some integer $k = 0, \ldots, N-1$. Since R^g is a pure reflection about an axis t, this may happen only if w and t form an angle of $k\pi/N$.

References

MONTGOMERY, D., AND ZIPPIN, L. 1955. *Topological transformation groups*, vol. 1. Interscience Publishers New York.