

## Elasticity Theory Basics

Lecture #7: 20 October 2003  
Lecturer: Denis Zorin  
Scribe: Adrian Secord, Yotam Gingold

### Introduction

This lecture is the summary of the basics of the elasticity theory. We will discuss the fundamental concepts of stress and strain, and the equations describing the deformation of an elastic object in equilibrium. We will focus on materials following the linear material law.

The linear material law can be thought of as a generalization Hooke's law  $f = -kx$  to 3D deformable objects. The most straightforward generalization would be to state that stretching or compressing an object in any direction results in an elastic force in the same direction which is proportional to the extension. However, this would match well only a small class of materials. In three dimensions in most cases a stretch in one direction changes the dimensions in the perpendicular directions. The two aspects of deformation are characterized using two constants: the Young's modulus, which corresponds to the constant in Hooke's law, and is denoted  $E$  or  $Y$ , and the Poisson's ratio, which is denoted  $\nu$ . The Poisson ratio is the ratio of stretch to perpendicular change. To formulate the equations of balance for elastic objects and define the linear material law more precisely, we start with a discussion of the counterparts of forces and displacements in the equations of elasticity: strain and stress tensors.

### Stress and Strain

There are two kinds of forces acting on a small volume of a body:

1. Volume forces

- $f$  is the volume force density;
- $dF_{vol} = f dV$  is the differential volume force, where  $dV$  is the differential volume element.

2. Surface forces

- $t(n)$  the vector surface force density, where  $n$  is the normal to the differential surface element  $dA$ ;
- $t(n)$  is also called the *stress vector*;
- $dF_{surf} = t(n)dA$  is the differential surface force, acting on the differential surface element  $dA$ .

## Stress tensor

As stated, the stress vector  $t(n)$  can have any form at all, but it turns out that it can be expressed in the special form  $t(n) = Tn$ , where  $T$  is called the Cauchy stress tensor.

**An aside on tensors.** 2D tensors are matrices which follow certain transformation rules when we change a coordinate system. Typically these rules follow from the definition of the quantity described by the matrix. In the case of the stress tensor, this can be seen as follows. Let  $A$  be the matrix representing a coordinate transformation  $x' = Ax$ . Under an arbitrary linear change of coordinates, normals are not generally transformed as points or displacement vectors. The correct transformation rule for normals can be derived if we keep in mind that if  $n$  is the normal to a plane  $P$  the transformed normal  $n'$  should remain perpendicular to the transformed plane  $P'$ . Then it is easy to derive that  $n' = A^{-T}n$ . Note that in the case of rigid transformations we have  $A^{-1} = A^T$ , so  $A^{-T} = A$ . On the other hand force vectors are transformed as all other vectors. Using these considerations, we have

$$\begin{aligned} t'(n) &= T'n' \\ At(n) &= T'A^{-T}n \\ t(n) &= A^{-1}T'A^{-T}n \end{aligned}$$

which implies that  $T = A^{-1}T'A^{-T}$  or  $T' = ATA^T$ . So the stress tensor is transformed using  $T' = ATA^T$ .

**Back to the stress tensor.** In general, the stress tensor is

$$T = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}, \quad (1)$$

which means that, for example,

$$T\hat{i} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{31} \end{pmatrix} \quad (2)$$

is the force if the normal to the surface area points along the  $x$ -axis.

We assume that the volume torques will balance, i.e. the volume forces create no net torque. This is generally assumed in elasticity. One can derive from this constraint that  $T$  is symmetric and has only six distinct elements.

**An aside on symmetric matrices.** As an aside, some properties of a general symmetric matrix  $S$  are

- The eigenvalues are always real (and as a consequence the eigenvectors have real components);
- $R^T S R$  is diagonal for some (rotation) matrix  $R$ , which means that  $S = R^T D R$ , where  $D$  is a diagonal matrix with the eigenvalues of  $S$  along its diagonal.

This means that we can always find some change of basis that transforms  $T$  into a nice diagonal form

$$T = \begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{pmatrix}. \quad (3)$$

The  $\sigma_i$ 's are called the principal stresses. Choosing these coordinates brings us closer to our goal of defining the generalization of Hooke's law for 3D materials. The second ingredient that we need is the measure of the deformation.

## Strain tensor

The strain tensor measures the change of distances between close points in the deformed state with respect to the distances in the undeformed state. Let  $\varphi : \Omega \rightarrow \mathbb{R}^3$  be the mapping from the reference domain  $\Omega$  to the deformed state. Compare the distance between two points that are close together in the reference frame,  $\|(x + dx) - x\|^2$ , with the distance between them in the deformed frame  $\|\varphi(x + dx) - \varphi(x)\|^2$ . We have

$$\|(x + dx) - x\|^2 = \|dx\|^2 = dx^T dx$$

in the reference frame and

$$\|\varphi(x + dx) - \varphi(x)\|^2 = (\varphi(x + dx) - \varphi(x))^T (\varphi(x + dx) - \varphi(x)) \quad (4)$$

in the deformed frame. For small deformations, we can approximate  $\varphi(x + dx)$  with  $\varphi(x) + \nabla\varphi dx$ , so (4) becomes

$$\begin{aligned} (\varphi(x + dx) - \varphi(x))^T (\varphi(x + dx) - \varphi(x)) &\approx (\nabla\varphi dx)^T (\nabla\varphi dx) \\ &= dx^T \nabla\varphi^T \nabla\varphi dx \end{aligned}$$

Aside: the gradient of a vector is a matrix. Specifically, if  $\varphi(z) = (\varphi_1(z), \varphi_2(z), \varphi_3(z))^T$ , then

$$\nabla\varphi = \begin{pmatrix} \nabla\varphi_1 \\ \nabla\varphi_2 \\ \nabla\varphi_3 \end{pmatrix}.$$

The quantity  $\nabla\varphi^T\nabla\varphi$  is called the Cauchy-Green deformation tensor. Finally, if we take the difference between the reference and deformed squared distances, we get

$$dx^T\nabla\varphi^T\nabla\varphi dx - dx^T dx = dx^T(\nabla\varphi^T\nabla\varphi - I)dx$$

The quantity  $\nabla\varphi^T\nabla\varphi - I$  is twice the *strain tensor*. Note that strain is non-dimensional.

Note that it is a matrix, and it is easy to derive its transformation rules, which end up the same as for the stress tensor. It follows directly from the definition that this tensor is symmetric:

$$\epsilon = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} \end{pmatrix},$$

We can also choose a coordinate system in which the strain is diagonal:

$$\epsilon = \begin{pmatrix} \epsilon_1 & & \\ & \epsilon_2 & \\ & & \epsilon_3 \end{pmatrix}.$$

The diagonal elements are called principal strains. This mathematical fact has important physical meaning: any deformation locally can be regarded as a simple stretch/compression along three perpendicular directions.

To get a better idea about the meaning of the principal strains, let's consider the change in length along a principal direction; according to the definition of strain, the new squared length is given by  $(dx_1, 0, 0)(2\epsilon + I)(dx_1, 0, 0)^T$ , i.e.

$$\begin{pmatrix} dx_1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2\epsilon_1 + 1 & & \\ & 2\epsilon_2 + 1 & \\ & & 2\epsilon_3 + 1 \end{pmatrix} \begin{pmatrix} dx_1 \\ 0 \\ 0 \end{pmatrix} = (2\epsilon_1 + 1)dx^2 \equiv dx_1'^2$$

Assuming small strains,  $\sqrt{2\epsilon_1 + 1} \approx 1 + \epsilon_1$ . So  $\frac{|dx_1'| - |dx_1|}{|dx_1|} = \epsilon_1$ , where the first quantity is the relative change in length in the  $x_1$  direction.

## Equations of balance

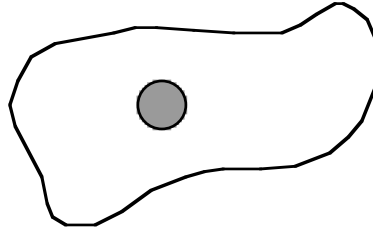
We will focus on elastostatic equations. Elasto-static equations are simpler to understand than elastodynamic; at the same time, it is relatively easy to generalize elastostatic equations.

These equations are derived from the equations of the balance of force and the balance of torques:

$$\begin{aligned} \sum f_i &= 0 && \text{sum of forces on body} \\ \sum r_i \times f_i &= 0 && \text{sum of moments of forces on body} \end{aligned}$$

and can be written either in the integral or differential form. We start with the the integral form.

We examine a small region of a deformed object  $dV$ .  $\partial dV$  denotes its surface.

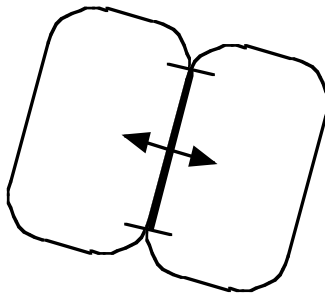


If  $f_V$  is the volume force density,

$$\int f_V dV + \underbrace{\int_{\partial dV} t(n) dS}_{\text{integrate over surface}} = \int f_V dV + \int_{\partial dV} T n dS \quad (5)$$

which must be zero to stay balanced.

Given two volumes in contact along surface element  $dA$ ,



the two surface forces with which the volumes act on each other are the same. In this way all internal surface forces cancel out.

From equation (5), summing over all small volumes into which we have partitioned our object, we get the balance of forces equation:

$$\int_V f_V dV + \int_{\partial V} T n dS = 0$$

Similarly, the torques need to stay balanced:

$$\int_V r \times f_V dV + \int_{\partial V} r \times T n dS = 0$$

To get the differential form of these equations, we use the Gauss Integral Theorem:

$$\int_{\partial V} T n dS = \int_V \text{div } T dV,$$

where  $(\text{div } T)$  is the divergence of  $T$ .

**An aside on tensor divergence.** For a vector function

$$f = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix},$$

divergence is scalar:

$$\text{div } f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}.$$

For a tensor, divergence is vector (sum over each column).

Returning to our two equations,

$$\int_V f_V dV + \int_{\partial V} Tn dS = 0 \quad (6)$$

$$\int_V r \times f_V dV + \int_{\partial V} r \times Tn dS = 0 \quad (7)$$

From equation (6),

$$\begin{aligned} \int_V (f_V + \text{div } T) dV &= 0 \\ \Rightarrow f_V + \text{div } T &= 0 \end{aligned}$$

The last equation is the differential form of the force balance. Recall that the symmetry of the stress tensor was a consequence of the torque balance, so the torque equation is captured by symmetry of the stress tensor and the boundary conditions.

The stress of the boundary is equal to the external stress:

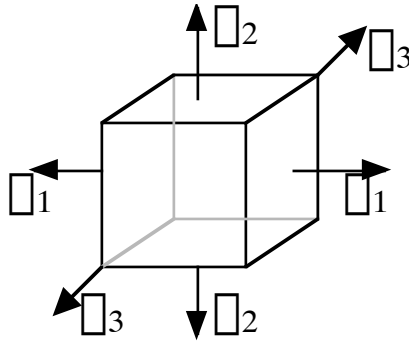
$$T = T_{ext} \text{ on } \partial V \text{ (stress continuity).}$$

An important fact to mention is that these equations are written with respect to the coordinates in the *deformed* object. In practice, it is much more convenient to use the coordinates in the undeformed object to write these equations, as typically the deformation is unknown and needs to be computed.

## Material laws, a.k.a. constitutive equations.

As we have mentioned in the beginning, we are primarily interested in materials following a generalization of Hooke's law. In general the elastic material behavior is captured by writing the strain as a function of stress.

We fix the coordinate system in which the stress tensor  $\Sigma$  is diagonal. In this coordinate system the forces acting on the elementary volume act along the coordinate axes; for each axis, the forces acting on opposite sides are equal in magnitude and opposite in direction.



In this coordinate system, the linear material law has the form

$$\epsilon_i = \frac{1}{E}\sigma_i - \frac{\nu}{E}(\sigma_j + \sigma_k) \quad (8)$$

where  $E$  is the Young modulus and  $\nu$  is the Poisson ratio. The first term in the formula reflects the assumption that the strain in a given direction depend linearly on the stress in the same direction. The second term takes into account the deformation due to the stresses in other directions, taking into account the degree to which the object can compress. For example, for incompressible objects  $\nu = 1/2$ , and the volume is preserved.

While this is a simple an intuitive way to write these equations, this requires using a coordinate system which might change from one point to another. In general, we need the form of these equations in an arbitrarily chosen coordinate system.

A simple way to arrive at such a form is to express everything in terms of tensor invariants.

**An aside on tensor invariants.** What are invariants of a symmetric 3D tensor? (In other words, functions of matrix elements that do not change when the coordinates change.) Eigenvalues can be regarded as invariants as we look at the whole set (the order may change as we change the coordinate system). It is more convenient to use symmetric functions of eigenvalues:

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 &= \text{Tr } \epsilon = \sum \epsilon_{ii} \\ \lambda_1 \lambda_2 \lambda_3 &= \det \epsilon \\ \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 &= \text{Tr } \epsilon^2 = \text{Tr} \begin{bmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \lambda_3^2 \end{bmatrix} = \frac{1}{2} [(\text{Tr } \epsilon)^2 - (\text{Tr } \epsilon^2)] \end{aligned}$$

Intuitively, one can think of the action of the deformation on a small sphere. The result is an ellipsoid with major axes lengths proportional to eigenvalues. Then

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 &= 3 \cdot \text{average cross section} \\ \lambda_1 \lambda_2 \lambda_3 &= \text{volume} \\ \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 &= 3 \cdot \text{length of diagonal} \end{aligned}$$

Equation (8) can be rewritten as

$$\epsilon_i = \frac{1 + \nu}{E} \sigma_i - \frac{\nu}{E} (\sigma_i + \sigma_j + \sigma_k).$$

Since  $\sigma_i + \sigma_j + \sigma_k = \text{Tr } \Sigma$ , we can write

$$\epsilon = \frac{1 + \nu}{E} \sigma - \frac{\nu}{E} (\text{Tr } \Sigma) I \quad (9)$$

This is the coordinate-independent form of the material linear law. Materials that follow this law are called St. Venant-Kirchhoff materials. Note that this law is more universal than it might seem: most isotropic elastic materials are quite well described by it. The reason for this is that as long as the strain is a function of stress, in most cases for small strains it is approximated well by the linear term in its Taylor series expansion. But strains tend to be small in most cases, excluding highly deformable objects such as rubber.

## Lamé constants

We can invert equation (9) to write  $\Sigma$  as a function of  $\epsilon$ . For this, we need to express the trace of stress in terms of the trace of strain, which can be done by taking the trace of both sides of the linear material law:

$$\text{Tr } \epsilon = \frac{1 + \nu}{E} \text{Tr } \Sigma - \frac{3\nu}{E} \text{Tr } \Sigma$$

Substituting the resulting expressions back, we obtain an expression of the form

$$\Sigma = \lambda(\text{Tr } \epsilon)I + 2\mu\epsilon.$$

Where the constant  $\lambda$  and  $\mu$  are called Lamé constants:

## Summary

$$f_V + \text{div } T = 0$$

and

$$\Sigma = \lambda(\text{Tr } \epsilon)I + 2\mu\epsilon$$

The stress tensors in these equations are denoted with different letters for a reason. The first equation is written with respect to the deformed body, and the constitutive law is written with variables specified in the original body coordinates. So these stress tensors are different: the second equation uses what is called the second Piola-Kirchhoff tensor and these two tensors are related by

$$\Sigma = (\det \phi)^{-1} \nabla \phi T \nabla \phi^T$$

Fortunately, the form of the equation of the balance of forces does not change its form when we switch to the undeformed body quantities.

The unknowns in these equations are the deformations (recall that the stress is defined using the gradient of the deformation function) and the stress tensor. The deformations are defined by three coordinate functions, and the stresses by six, so we have a total of nine unknown functions. The force balance results in three equations with the constitutive law providing additional six, so now the system of equations is complete. Of course, this simple counting argument does not prove that the solution exists or is unique.