Transformations

Examples of transformations:

- Translation
- Rotation
- Scaling
- Shear

More examples:

- Reflection with respect to the y-axis
- Reflection with respect to the origin

Linear transformations: take straight lines to straight lines.
All of the examples are linear.
Affine transformations: take parallel lines to parallel lines.
All of the examples are affine,
an example of linear non-affine is perspective projection.
Orthogonal transformations: preserve distances,
move all objects as rigid bodies.
rotation, translation and reflections are affine.

Transformations and matrices

Any affine transformation can be written as

\[
\begin{pmatrix}
  x' \\
  y'
\end{pmatrix} = \begin{pmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{pmatrix} \begin{pmatrix}
  x \\
  y
\end{pmatrix} + \begin{pmatrix}
  b_1 \\
  b_2
\end{pmatrix}
\]

Images of basis vectors under affine transformations:

- \[ e_x = \begin{pmatrix}
  1 \\
  0
\end{pmatrix} \]
  (column form of writing vectors)
- \[ A e_x = \begin{pmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{pmatrix} \begin{pmatrix}
  1 \\
  0
\end{pmatrix} = \begin{pmatrix}
  a_{11} \\
  a_{21}
\end{pmatrix} \]
- \[ A e_y = \begin{pmatrix}
  a_{12} \\
  a_{22}
\end{pmatrix} \]

Matrices of some transformations:

- Shear
  \[ \begin{pmatrix}
  1 & s \\
  0 & 0
\end{pmatrix} \] scale by factor s
- Rotation
  \[ \begin{pmatrix}
  \cos \alpha & -\sin \alpha \\
  \sin \alpha & \cos \alpha
\end{pmatrix} \] rotation
- Reflection with respect to the origin
  \[ \begin{pmatrix}
  -1 & 0 \\
  0 & -1
\end{pmatrix} \]
- Reflection with respect to the y-axis
  \[ \begin{pmatrix}
  -1 & 0 \\
  0 & 1
\end{pmatrix} \]
Determinant

Area of a parallelogram

Area of the shaded parallelogram:

\[
\text{area} = |v \times w| \sin \alpha
\]

Determinant

Determinant of a transformation matrix = area of the image of a unit square.

\[
\text{area} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}
\]

3D vectors

Same as 2D (directed line segments with position ignored), but we have different properties.

In 2D, the vector perpendicular to a given vector is unique (up to a scale).
In 3D, it is not.

Two 3D vectors in 3D can be multiplied to get a vector (vector or cross product).

Dot product works the same way, but the coordinate expression is

\[
(v, w) = v_x w_x + v_y w_y + v_z w_z
\]

Vector (cross) product

\[
v \times w \text{ has length } |v||w|\sin \alpha
\]

= area of the parallelogram with two sides given by v and w, and is perpendicular to the plane of v and w.

\[
(v + w) \times u = v \times u + w \times u
\]

Direction (up or down) is determined by the right-hand rule.

\[
v \times w = -w \times v
\]

Unlike a product of numbers or dot product, vector product is not commutative!

Vector product

Coordinate expressions

\[
v \times w \text{ is perpendicular to } v, \text{ and } w:
\]

\[
(u, v) = 0 \quad (u, w) = 0
\]

the length of u is \(|v||w|\sin \alpha |)

\[
(u, u) = |v||w|^2 \sin^2 \alpha = |v||w|^2 (1 - \cos^2 \alpha)
\]

\[
= |v||w||v||w| - (v, w)
\]

Solve three equations for \(u_x, u_y, u_z\)

Vector product

Physical interpretation: torque

\[
\text{torque} = r \times F
\]

phase of rotation
force F
displacement r
Vector product

Coordinate expression:

\[
\begin{vmatrix}
\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\
\mathbf{v}_x & \mathbf{v}_y & \mathbf{v}_z \\
\mathbf{w}_x & \mathbf{w}_y & \mathbf{w}_z
\end{vmatrix}
= \begin{vmatrix}
\mathbf{v}_x & \mathbf{v}_y \\
\mathbf{w}_x & \mathbf{w}_y
\end{vmatrix} \cdot \begin{vmatrix}
\mathbf{v}_z \\
\mathbf{w}_z
\end{vmatrix}
\]

Notice that if \( \mathbf{v}_x = \mathbf{w}_x = 0 \), that is, vectors are 2D, the cross product has only one nonzero component (\( \mathbf{v}_y \)) and its length is the determinant

\[
\begin{vmatrix}
\mathbf{v}_x & \mathbf{v}_y \\
\mathbf{w}_x & \mathbf{w}_y
\end{vmatrix}
\]

Vector product

More properties

\[
(a, b \times c) = b(ac) - c(ab)
\]

\[
(a \times b, c \times d) = (a, c)(b, d) - (b, c)(a, d)
\]

Plane equations

Implicit equation: \( (q-p, n) = 0 \), exactly like line in 2D!

Parametric equation: 2 parameters \( t_1, t_2 \)

\[
q(t_1, t_2) = v_1 t_1 + v_2 t_2, \text{ where } v_1 \text{ and } v_2 \text{ are two vectors in the plane.}
\]

\[
v_1 \times v_2 = n
\]

Intersecting a line and a plane

Same old trick: use the parametric equation for the line, implicit for the plane.

\[
(p_1 + vt - p_2, n) = 0
\]

\[
t' = \frac{(p_1 - p_2, n)}{(v, n)} \quad \text{Do not forget to check for zero in the denominator!}
\]