Manifold Based Construction of Arbitrarily Smooth Surfaces

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The Problem
Smooth Surface Construction

Main Approaches

- Stitching polynomial patches
  - Need the match patches on boundaries
  - Complexity increases with smoothness

- Subdivision
  - Good visual quality
  - \( C^2 \) away from extraordinary vertices
Manifold Based Approach

- Introduced by Grimm & Hughes 1995
- Idea:
  - Parameterize over overlapping patches
  - Blend local geometries to get global geometry
- Solves problem of “stitching”
- Also the discontinuities near extra ordinary vertices
Why Manifolds?

Allows us to have:

- Any prescribed smoothness including $C^¥$ with explicit nonsingular parameterizations of same smoothness
- At least 3-flexible surfaces
  - Prescribe derivatives to avoid artificial flat spots
- Linear dependence on control points
- Fixed size local support for basis functions
- Good visual quality … …simultaneously!
Introduction: Manifolds

Manifold $M$  

Charts $(C_i, j_i)$

$\begin{align*}
&\text{If transition map } t_{ji} \in C^k \text{ then } M \text{ is a } C^k \text{ manifold} \\
&(C_i, j_i) = M
\end{align*}$
Introduction: Embedding

- \( f_i^l : C_i \rightarrow \mathbb{R}^3 \) defines geometry locally per chart
- Use partition of unity to define global geometry
  \( (\sum_i w_i \circ j_i^{-1} = 1) \)
- Geometry evaluated per chart \( C_i \):
  \[
  f_i(x) = \sum_{j: \varphi_i(x) \in \varphi_j(C_j)} w_j(t_{ji}(x)) f_j^l(t_{ji}(x))
  \]
- 3 components: transition maps \( t_{ji} \)
  pou \( w_i \)
  geometry functions \( f_i^l \)

"If all components \( C^{\text{inf}} \) then surface \( C^{\text{inf}}"
Charts & Transition Maps

\[ j_i^{-1} = c_i \circ L_i \]

- \( L_i \): maps faces bilinearly to plane
- \( c_i \): wedge \( \rightarrow \) conformal \( l_k \)
  image of square

\[ S_1 \xrightarrow{\text{transition map}} S_2 \]

\[ D_1 \xrightarrow{\text{transition map}} D_2 \]
$j_i^{-1} = c_i \hat{e} L_i$

- $L_i$: maps faces bilinearly to plane
- $c_i$: wedge $\rightarrow$ conformal image of square

$c_i = g_k \circ l_k$

$g_k = z^{(4/k)}$

$$l_k = \begin{bmatrix}
\cos(\pi / 4) & 0 \\
\cos(\pi / k) & 0 \\
0 & \sin(\pi / 4) \\
0 & \sin(\pi / k)
\end{bmatrix}$$
Partition of Unity

\[ \eta(t) \in C^\infty \]

\[ \eta(t) + \eta(1 - t) = 1 \]

\[ w_i(m) = f_i(\mathbf{j}_i^{-1}(m)) \]
Geometry

- Polynomial fit based on least squares
- Use subdivision to generate points to fit surface
  - Subdivide twice & gather points in interior
    - $12k+1$ 3D pts: $s_i$

- In the chart
  - $12k+1$ chart pts
Geometry

- Want $f$ s.t. $\min \left( |f(x_i) - s_i| \right)$
- Least squares polynomial fitting

Degree: $d = \min(14, K+1)$,
Nr of monomials with degree $\leq d$: $N = \frac{(d+1)(d+2)}{2}$
Nr of data points: $M = 12k+1$

Define $U = \begin{bmatrix} N \\ M \end{bmatrix}$

Minimizing $\|Ua-s\|^2$

Let $a = Bs$ . Then minimize $\|UBs-s\|^2 \rightarrow UB = I \rightarrow a = U^+s$
What has been done?

- Implementation for $C^\infty$ closed surfaces by Lexing Ying as part of PhD thesis

(To appear, SIGGRAPH 04)
Contributions

- $C^{\infty}$ surfaces with boundaries
  - Smooth Boundaries
  - Convex and Concave Corners

- $C^k$ surfaces based on spline fitting
  - Closed Surfaces
  - Smooth Boundaries
  - Convex and Concave Corners
  - $C^k$ Blending Functions
\( C^{\text{inf}} : \) Smooth Boundaries

- Charts and Transition Maps
  - Charts:
    - Domain half curved star
    - Overlap 1 wedge
  - Transition maps:
    \[
    j_i^{-1} = c_i \circ L_i, \quad c_i = g_k \circ l_k
    \]

\[
 g_k = z \left( \frac{4}{2^k} \right) = z \left( \frac{2}{k} \right) \]

\[
 l_k = \begin{bmatrix}
 \cos(\pi/4) & 0 \\
 \cos(\pi/2k) & 0 \\
 0 & \sin(\pi/4) \\
 0 & \sin(\pi/2k) 
\end{bmatrix}
\]

- Partition of Unity - Exactly the same (per wedge)
$C^{\inf}$: Smooth Boundaries

- **Geometry**
  - Subdivision: $M = 12k+4$
  - Least Squares – 2 Approaches

- **Global system**
  
  \[ f = \sum_{j=0}^{n-1} a_j p_j \]
  
  \[ a = Bs \text{ s.t. } BU = I \quad \Rightarrow \quad a = U^+ s \]

- **Independent Boundary**

\[
BU = \begin{pmatrix}
    B_1 & B_2 \\
    0 & B_3
  \end{pmatrix}
\begin{pmatrix}
    12k-3 \\
    7
  \end{pmatrix}
\begin{pmatrix}
    N' & Nu
  \end{pmatrix}
= \begin{pmatrix}
    1 & 0 \\
    0 & 1
  \end{pmatrix}
\begin{pmatrix}
    B_1 U_1 = I \\
    B_1 U_2 + B_2 U_3 = 0 \\
    B_3 U_3 = I
  \end{pmatrix}
\]
Results
Our Surface vs. Catmull-Clark
Global vs. Independent Boundary
$C^{\infty}$: Corners

- Concave and Convex Corners need to be treated separately:
  
  No smooth non-degenerate map between

  ➔ Need two different sets of charts and transition functions

  ➔ Use the modified concave corner rules (the flatness parameter from Biermann, et al) in subdivision step
$C^\infty$: Corners

**Convex Charts**

Transition Maps

$$g_k = z^{(4/4)k} = z^{(1/k)}$$

$$l_k = \begin{bmatrix} \frac{\cos(\pi/4)}{\cos(\pi/4k)} & 0 \\ \sin(\pi/4) & \sin(\pi/4k) \end{bmatrix}$$

**Concave Charts**

Transition Maps

$$g_k = z^{(3*4/4)k} = z^{(3/k)}$$

$$l_k = \begin{bmatrix} \frac{\cos(\pi/4)}{\cos(3\pi/4k)} & 0 \\ \sin(\pi/4) & \sin(3\pi/4k) \end{bmatrix}$$
**C^\text{inf}:** Corners (Convex & Concave)

**Geometry:**

\[ M = 12k + 4 \text{ data pts} \]

- **Global System**— exactly the same as \( C^\text{inf} \) boundary
- **Independent Boundary:**

\[
\begin{align*}
\mathbf{B} \mathbf{U} &= \mathbf{N}' \\
N'(u,v,1) &= \begin{bmatrix} B1 & B2 & 12k-3 \\ 0 & B3 & 7 \end{bmatrix} \\
N'(u,v,1) &= \begin{bmatrix} U1 & U2 & 7 \\ 0 & U3 & 12k-3 \end{bmatrix}
\end{align*}
\]

\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\]

\[
\begin{aligned}
B_1 U_1 &= 1 \\
B_1 U_2 + B_2 U_3 &= 0 \\
B_3 U_3 &= 1
\end{aligned}
\]
Results: Convex

Val 2

Val 5

Val 3
Results: Concave

Indep. Boun with flatness

Both convex and concave corners

Val 3
Global
No flatness
Catmull-Clark vs. Our Surface
$C^k$ Embedding

Same manifold different embedding

Why do we want $C^k$?

Higher derivatives of $C^\infty$ surfaces too high
\( C^k \) Embedding

- All charts and transition maps are the same as those of \( C^{\text{inf}} \) surfaces.

- The only changes are:
  - Partition of Unity
    - Spline basis functions of degree \( k+1 \)
  - Geometry
$C^k : \text{Closed Surfaces}$

- 12$k$+1 data pts = $S$
- Fit a grid around
  
  \[ nrgrid^2 \sim 12k+1 \]

  \[ f(x, y, c) = \sum_{i=-k}^{p+k} \sum_{j=-k}^{q+k} N_i(x)N_j(y)p_{ij} \]

  \[ S = MP \]

Solve for ctrl points:

\[ P = M^+ S \]

($M$ is 12$k$+1 x (nrgrid)$^2$)
C^k : Problems and Solutions

Problem 1:
Since data points not uniformly placed, some control points (near corners) may be way too off.

Solution A:
Use an extra ring of points.
(Fit to 20k+1 pts s.t. bad behavior will happen farther away from the part of the chart that we need)
Problem 1:
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C^k : Problems and Solutions

Problem 1:
Since data points not uniformly placed, some control points (near corners) may be way too off.

Solution B:
Add “smoothness” constraints
For each interior grid point, use “weighted” 2^{nd} order finite differences:

```
  1   -2   1
-2   4  -2
  1   -2   1
```
Problem 2:
For higher $k$, number of “extra” grid points may be too large $\rightarrow$ underdetermined system

Solution:
In this case, the whole chart is contained in one polynomial patch.
$\Rightarrow$ Already $C^{\infty}$. 
**Problem 2:**

For higher k, number of “extra” grid points may be too large → underdetermined system

**Solution:**

In this case, the whole chart is contained in one polynomial patch. ⇒ Already $C^{\infty}$.

Lower the degree (k) until the system becomes over-determined again.
The geometry changes slightly with increasing degree but the surface quality is maintained
Polynomials

Max Value 2.581 3.71911 5.65685 2055.46

Splines(C⁵)

Max Value 1.89508 1.79864 3.98121 1059.81
Some more complex examples...
$C^k : \text{Smooth Boundary & Corners}$

- Only have the “global system” for all the following so far.
  - Boundary
  - Concave
  - Convex

In each case, fit the relevant chart onto a grid and repeat same procedure
Results: Boundary

C-5, Val 6

Val 10
Results: Convex
Results: Concave
What’s Next

- Interior ctrl point independent boundary and corners for $C^k$ surfaces
- Efficiency of the implementation needs to be improved (splines)
- Issues that need to be resolved:
  - $C$-Inf, concave corner, valance 2
  - Convex vertices for splines of higher degree

\[\frac{2\pi}{3}\]
\[\frac{3\pi}{4}\]
Conclusions

- $C^{\infty}$ surface construction based on manifolds can be extended to surfaces with boundaries and corners.
- $C^k$ surface construction with/out boundaries and corners based on manifolds possible through use of splines.
- If $C^k$ is needed instead of $C^{\infty}$, splines work well. However:
  - Implementation is more complicated
  - Tuning plays a big role
- Visual quality of splines better than that of polynomial near higher valence vertices