Optimal twist vectors as a tool for interpolating a network of curves with a minimum energy surface

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Abstract. This paper provides a new answer to the old problem of specifying the mixed partial derivatives (MPDs) or 'twist vectors' at the grid points for an interpolating surface over a rectangular network of curves. An algorithm is presented for finding the MPDs that minimizes a generalized energy integral over the entire surface. The integrand may be any quadratic form in the second partial derivatives of the surface. This results in a surface design technique for interpolating over a network of curves by automatically selecting the optimal twist vectors at the grid points.

Introduction

Given a mesh of cubic spline curves, \( C_j(u) \) over the knots \( u_1 < \cdots < u_m \) and \( D_i(v) \) over \( v_1 < \cdots < v_n \), with intersections at \( C_j(u_i) = D_i(v_j) \), we sometimes seek an interpolating piecewise-bicubic surface \( R(u, v) \) which matches the curves, that is \( R(u, v_j) = C_j(u) \) and \( R(u_i, v) = D_i(v) \). These requirements do not determine the surface uniquely, and some surfaces are 'better' than others in some sense. The curves are uniquely defined by their points and tangents \( C_j(u_i), D_i(v_j), C_j'(u_i), D_i'(v_j) \) at the intersection points. The \((i, j)\)th patch of the surface is uniquely defined by the points \( R \) and partial derivatives \( R_u, R_v, R_{uv} \) at the four corners \((u_i, v_j), (u_{i+1}, v_j), (u_i, v_{j+1}) \) and \((u_{i+1}, v_{j+1})\). Matching the curves at all patches is achieved if \( R(u_i, v_j) = C_j(u_i), R_u(u_i, v_j) = C_j'(u_i) \) and \( R_v(u_i, v_j) = D_i'(v_j) \) for all \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \), but the so called MPDs (mixed partial derivatives, or 'twist vectors') \( R_{uv}(u_i, v_j) \) remain at our disposal. In the past, these MPDs have sometimes been viewed as a liability, to be taken care of in order to 'stay out of trouble' ('trouble' meaning unwanted wrinkles in the surface). Their 'correct' values were to be estimated [Akima '78, Barnhill et al. '78], or, for lack of better information, they were set to zero [Dube '75]. A natural alternative is to seek the twist vectors that will minimize some energy integral over the entire surface. Hagen and Schulze [Hagen & Schulze '87] used the sum of the squares of the principal curvatures as the energy integrand, with a variational approach. Dube [Dube '75] maximizes another functional, but his constraints are points, not curves. See [Barnhill et al. '88] for a recent survey on the twist vector problem. Here, in turn, we treat the twist vectors as an asset, providing additional control over the shape.

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of the surface. An algorithm is presented for finding the MPDs that minimize a user-defined 'energy' integral over the surface. Our energy is the integral of any positive-definite quadratic form in the second partial derivatives over the entire surface, providing a very general form for the energy functional. Our approach is somewhat similar to [Hagan & Schulze '87], but we handle a more general energy functional, and we minimize it simultaneously over the entire multi-patch surface, without making any assumptions about it. Our resulting surface is bicubic (rather than biquintic).

Why quadratic forms?

Minimizing the exact elastic energy is computationally too difficult to be used as a design tool, even for curves. It is much more so for surfaces, where the physical model is much more complex. When we design spline curves we usually approximate the energy integrand, which is the square of the second derivative. The approximation is good as long as the first derivative is small [Faux & Pratt '79]. We may improve our approximation by breaking our curve into pieces, and using local coordinates in each piece where the derivative remains small [Fowler & Wilson '66]. In any case, we end up minimizing the integral of a quadratic function of the second derivative of the curve. It is therefore natural to expect an integral of some quadratic form in the second partial derivatives as the approximation of the energy integrand for surfaces. A general quadratic form in the second partial derivatives of the surface is

\[ f(x) = x^T Q x \]  

(1)

where \( x = [x_{uu}, x_{uv}, x_{v}, y_{uu}, y_{uv}, y_{v}, z_{uu}, z_{uv}, z_{v}]^T \) and \( Q \) is the square symmetric coefficient matrix. The energy functional minimized to construct the surface is:

\[ E = \int \int (x^T Q x) \, du \, dv. \]  

(2)

Minimizing \( E \) yields an optimal set of values for the twist vectors for the 'best' bicubic surface interpolating over the given network of cubic curves. What is 'best' depends on our criteria. Suppose, for example, that we wish to minimize the functional \( E = \int (K_1^2 + K_2^2) \, d\sigma \) where \( K_1 \) and \( K_2 \) are the principal curvatures of the surface and \( d\sigma \) is the surface area measure. This curvature functional can be written in terms of first and second surface derivatives if the principal curvatures are rewritten in terms of Gaussian and mean curvatures:

Gaussian Curvature = \( K = K_1 K_2 \),

Mean Curvature = \( H = (K_1 + K_2) / 2 \).

The energy integrand then becomes

\[ K_1^2 + K_2^2 = (2H)^2 - 2K. \]  

(3)

Now, if we consider the first partial derivatives of each surface patch constant over the patch by taking their average over the four corners of the patch, then the right-hand side of equation (3) becomes a quadratic form in the second partial derivatives of the surface.

Another example where our general quadratic form becomes handy is the thin plate spline surface \( z(x, y) \). The energy integral for elastic deflections of a thin rectangular plate is [Timoshenko & Woinowsky-Krieger '59]:

\[ E = \int \int [z_{xx}^2 + z_{yy}^2 + 2\nu \, z_{xx} z_{yy} - 2(1 - \nu) \, z_{xy}^2] \, dx \, dy \]

where \( \nu \) is Poisson's ratio (a constant).
Different choices of the entries of the coefficient matrix \( Q \) will result in different energy functionals that can be used for surface construction depending on particular design needs. For example, consider a surface design situation in which smoothness in one direction is more important than in other directions. This occurs in practice, for instance, in the design of an aircraft wing surface, where smoothness of the surface is critical in the airflow direction. If \( u \) and \( v \) are the parameters of the surface and \( u \) is approximately the airflow direction, then the entries of the coefficient matrix \( Q \) of the quadratic form should be chosen to make the energy integrand a function of the partial derivatives in \( u \) only. One example of such an energy functional is

\[
E = \int \int \left( x_{uu}^2 + y_{uu}^2 + z_{uu}^2 \right) \, du \, dv.
\]

An evaluation and comparisons of different forms of the quadratic energy functional for surface design is beyond the scope of this paper. Here, the emphasis is on the general framework for computing the interpolating surfaces once we have selected our design criteria and expressed them as an energy functional.

The mathematics involved in computing the optimal MPD's for a surface designed by the energy integral (2) is conceptually simple. The energy is a quadratic form in those variables, resulting in a set of linear equations as the optimality conditions. The details, however, are tedious; we will encounter six-index terms even before we reach full generality. We therefore start with the special case of designing scalar bicubic polynomials, and then gradually advance towards the general parametric bicubic surfaces. These are described in detail in the next two subsequent sections.

Scalar bicubic polynomials

In this section, we consider the problem of designing a scalar bicubic polynomial, by minimizing an energy integral of a positive definite quadratic form in the second partial derivatives of the polynomial.

Let us proceed with minimizing the energy monomial

\[
E = \int_0^1 \int_0^1 r_{uu}^2 \, du \, dv
\]

of a scalar bicubic polynomial \( r(u, v) \), where \( r(k, l) \), \( r_u(k, l) \) and \( r_v(k, l) \) are prescribed at the corners \( k, l \in \{0, 1\} \) of the unit square. \( r \) can be expressed in terms of its corner values as [Dodd et al. '83, p.204]:

\[
r(u, v) = \begin{bmatrix}
\alpha_0(u) \\
\alpha_1(u) \\
\beta_0(u) \\
\beta_1(u)
\end{bmatrix} \begin{bmatrix}
r(0, 0) & r(0, 1) & r_u(0, 0) & r_u(0, 1) \\
r(1, 0) & r(1, 1) & r_u(1, 0) & r_u(1, 1) \\
r_u(0, 0) & r_u(0, 1) & r_{uu}(0, 0) & r_{uu}(0, 1) \\
r_u(1, 0) & r_u(1, 1) & r_{uu}(1, 0) & r_{uu}(1, 1)
\end{bmatrix} \begin{bmatrix}
\alpha_0(v) \\
\alpha_1(v) \\
\beta_0(v) \\
\beta_1(v)
\end{bmatrix}
\]

where

\[
\alpha_0(t) = 1 - 3t^2 + 2t^3, \quad \alpha_1(t) = 3t^2 - 2t^3,
\]

\[
\beta_0(t) = t - 2t^2 + t^3, \quad \beta_1(t) = -t^2 + t^3
\]

are the cubic blending functions for \( [0, 1] \).
If we write the MPD-independent part as:

\[
u(u, v) = \begin{bmatrix}
\alpha_0(u) \\
\alpha_1(u) \\
\beta_0(u) \\
\beta_1(u)
\end{bmatrix}^T \begin{bmatrix}
r(0, 0) & r(0, 1) & r_e(0, 0) & r_e(0, 1) \\
r(1, 0) & r(1, 1) & r_e(1, 0) & r_e(1, 1) \\
r_u(0, 0) & r_u(0, 1) & 0 & 0 \\
r_e(1, 0) & r_e(1, 1) & 0 & 0
\end{bmatrix} \begin{bmatrix}
\alpha_0(v) \\
\alpha_1(v) \\
\beta_0(v) \\
\beta_1(v)
\end{bmatrix}
\]

then

\[r(u, v) = u(u, v) + \sum_{k,l \in \{0,1\}} \beta_k(u) \beta_l(v) r_{u_l}(k, l),\]

and we seek the minimum of

\[E = \int_0^1 \int_0^1 \left( a_{uu}(u, v) + \sum_{k,l \in \{0,1\}} \beta_k''(u) \beta_l''(v) r_{u_l}(k, l) \right)^2 \, du \, dv.\]

The necessary conditions

\[\frac{\partial E}{\partial r_{u_l}(s, t)} = 0 \quad \text{for} \quad s, t \in \{1, 0\}\]

yield four linear equations:

\[\sum_{k,l \in \{0,1\}} g(k, l, s, t) r_{u_l}(k, l) = h(s, t), \quad s, t \in \{0, 1\}\]

where

\[g(k, l, s, t) = 2 \int_0^1 \int_0^1 \beta_k''(u) \beta_l''(u) \beta_v''(u) \beta_v(v) \, du \, dv\]

and

\[h(s, t) = -2 \int_0^1 \int_0^1 a_{uu}(u, v) \beta_s''(u) \beta_t''(v) \, du \, dv.\]

A similar approach for the energy integrand \(r_{uu''} vv\) leads to

\[g(k, l, s, t) - \int_0^1 \int_0^1 (\beta_k''(u) \beta_l(u) \beta_v''(u) \beta_v(v)) + \beta_k(u) \beta_v''(u) \beta_v''(u) \beta_v(v)) \, du \, dv\]

and

\[h(s, t) = -\int_0^1 \int_0^1 (a_{uu}(u, v) \beta_s(u) \beta_t''(v) + a_{vv}(u, v) \beta_v''(u) \beta_v(v)) \, du \, dv.\]

Stepping further to a general quadratic form in the second derivatives of \(r\), we index the corners of the square by pairs of numbers in \(K = \{0, 1\}\), and the various derivatives by (single) indices in the set \(D = \{uu, uv, vu\}\). The energy is:

\[E = \sum_{d,e \in D} w(r_d, r_e) \int_0^1 \int_0^1 r_d r_e \, du \, dv, \quad \text{where} \quad w(r_e, r_d) = w(r_d, r_e).\]

Following the previous discussion, the energy integral is

\[E = \sum_{d,e \in D} w(r_d, r_e) \int_0^1 \int_0^1 \left( a(u, v) + \sum_{k,l \in K} \beta_k(u) \beta_l(v) r_{u_l}(k, l) \right)_d \times \left( a(u, v) + \sum_{k,l \in K} \beta_k(u) \beta_l(v) r_{u_l}(k, l) \right)_e \, du \, dv,\]
yielding four optimality equations $\frac{\partial E}{\partial r_{ul}}(s, t)$:

$$\sum_{k,l \in K} g(k, l, s, t) r_{ul}(k, l) = h(s, t) \quad (s, t \in K)$$

where

$$g(k, l, s, t) = \sum_{d, e \in D} w(r_d, r_e) \int_0^1 \int_0^1 (\beta_s(u) \beta_i(v))_d (\beta_k(u) \beta_l(v))_e + ((\beta_s(u) \beta_l(v))_d (\beta_k(u) \beta_i(v)))_e \, du \, dv$$

and

$$h(s, t) = -\sum_{d, e \in D} w(r_d, r_e) \int_0^1 \int_0^1 (\beta_s(u) \beta_i(v))_d a_e(u, v) + (\beta_s(u) \beta_i(v))_e a_d(u, v) \, du \, dv.$$

**Parametric bicubic surfaces**

We can now generalize the results of the previous section to three-dimensional bicubic surfaces of the form $R(u, v) = (x(u, v), y(u, v), z(u, v))$. In this case, we need another index set $C = \{x, y, z\}$. The general monomial is of the form $p_d q_e$ with $p, q \in C$ and $d, e \in D$, and the energy integrand is

$$\sum_{d, e \in D} \sum_{p, q \in C} w(p_d, q_e) p_d q_e.$$

The MPD-free polynomial $a(u, v)$ was originally defined in terms of prescribed corner values of $r, r_s$, and $r_t$, so now we have three separate polynomials $a^x(u, v)$, $a^y(u, v)$, and $a^z(u, v)$ for the three coordinate functions, and the energy integral is:

$$E = \sum_{p, q \in C} \sum_{d, e \in D} w(p_d, q_e) \int_0^1 \int_0^1 \left( a^p + \sum_{k,l \in K} \beta_k(u) \beta_l(v) p_{uw}(k, l) \right)_d \times \left( a^q + \sum_{k,l \in K} \beta_k(u) \beta_l(v) q_{uw}(k, l) \right)_e \, du \, dv.$$

For each $c \in C$ and $s, t \in \{0, 1\}$ we have an equation

$$0 = \frac{\partial E}{\partial c_{uc}}(s, t) = \sum_{d, e \in D} \left( \sum_{q \in C} w(c_d, q_e) \int_0^1 \int_0^1 (\beta_s(v) \beta_i(v))_d \times \left( a^q + \sum_{k,l \in K} \beta_k(u) \beta_l(v) q_{uw}(k, l) \right)_e \, du \, dv + \sum_{p \in C} w(p_d, c_e) \int_0^1 \int_0^1 (\beta_s(u) \beta_i(v))_e \times \left( a^p + \sum_{k,l \in K} \beta_k(u) \beta_l(v) p_{uw}(k, l) \right)_d \, du \, dv \right).$$
In the second term we may change the summation index \( p \) to \( q \), and interchange the indices \( d \) and \( e \) (since we are dealing with the sum over \( d, e \in D \)) to obtain:

\[
0 = \sum_{d,e \in D} \left( \sum_{q \in C} \left( w(c_d, q_e) \int_0^1 \int_0^1 (\beta_s(u) \beta_l(v))_d \times \left( a^q + \sum_{k,l \in K} \beta_k(u) \beta_l(v) \, q_{uv}(k,l) \right)_e \right) \right) du \, dv + w(q_e, c_d) \int_0^1 \int_0^1 (\beta_s(u) \beta_l(v))_d \times \left( a^q + \sum_{k,l \in K} \beta_k(u) \beta_l(v) \, q_{uv}(k,l) \right)_e \, du \, dv.
\]

Finally, recalling that \( w(q_e, c_d) = w(c_d, q_e) \) we obtain identical terms. Our linear equations are:

\[
\sum_{q \in C} \sum_{k,l \in K} g(s, t, c, k, l, q) \, g_{uv}(k,l) = h(s, t, c) \quad \text{for} \quad (s, t \in K, c \in C),
\]

where

\[
g(k, l, q, s, t, c) = \sum_{q \in C} \sum_{d,e \in D} w(c_d, q_e) \int_0^1 \int_0^1 (\beta_s(u) \beta_l(v))_d (\beta_k(u) \beta_l(v))_e \, du \, dv,
\]

and

\[
h(s, t, c) = - \sum_{q \in C} \sum_{d,e \in D} w(c_d, q_e) \int_0^1 \int_0^1 a^q (\beta_s(u) \beta_l(v))_d \, du \, dv.
\]

So far we have been fitting a simple mesh of four curves over the unit square. In the more general case we have an energy integrand

\[
E_{ij} = \int_{i}^{i+1} \int_{j}^{j+1} \sum_{p,q \in C} \sum_{d,e \in D} w(p_d, q_e) \, p_{dq} \, du \, dv
\]

for each one of the squares in a more general mesh, defined by \( u-i, v-j \) for \( 0 \leq i \leq m, 0 \leq j \leq n \), and we seek the minimum of \( E = \sum_i \sum_j E_{ij} \). For each coordinate \( q \in \{x, y, z\} \) we are given the values to \( q(i, j), q_u(i, j) \) and \( q_v(i, j) \), and the MPDs \( q_{uv}(i, j) \) are the variables of the optimization problem \( \min E \). \( \partial E_{ij}/\partial q_{uv}(u_i+v_j, u_{i+1}+v_{j+1}) \) obviously vanishes if \( s \in \{0,1\} \) or \( t \in \{0,1\} \), since the expression of the energy integrand at the \((i, j)\)th patch involves only MPDs at its four corners. The situation is similar to that of \( \partial E/\partial r_{uv}(s, t) \) in the simple single-patch case, but some changes are required. The MPD-free term \( a_{uu}(u, v) \) needs to be labeled, since it depends on the geometric constraints of the \((i, j)\)th patch. The blending functions \( a_s(u), a_l(v) \) (for \( s, t \in [0,1] \)) are replaced by \( a_s(u-i) \) and \( a_l(v-j) \), respectively. With these changes, we have for \( s, t \in \{0,1\} \) and \( c \in \{x, y, z\} \):

\[
\frac{\partial E_{ij}}{\partial c_{uv}(i+j, s+t)} = \sum_{k,l \in K} \sum_{q \in C} g_{ij}(k, l, q, s, t, c) \, q_{uv}(i+k, j+l) - h_{ij}(s, t, c)
\]

where

\[
g_{ij}(k, l, q, s, t, c) = \sum_{q \in C} \sum_{d,e \in D} w(c_d, q_e) \int_{i}^{i+1} \int_{j}^{j+1} (\beta_s(u-i) \beta_l(v-j))_d \times (\beta_k(u-i) \beta_l(v-j))_e \, du \, dv
\]

\[
h(s, t, c) = - \sum_{q \in C} \sum_{d,e \in D} w(c_d, q_e) \int_{i}^{i+1} \int_{j}^{j+1} a^{q,ij}(u, v) \beta_s(u-i) \beta_l(v-j) \, du \, dv,
\]
and \( a^{i,j}(u, v) = F(u - 1)^T Q^{ij} F(v - j) \) with \( F(t) = [\alpha_0(t), \alpha_1(t), \beta_0(t), \beta_1(t)] \) and

\[
Q_{ij} = \begin{bmatrix}
q(i, j) & q(i, j + 1) & q_0(i, j) & q_0(i, j + 1) \\
q(i + 1, j) & q(i + 1, j + 1) & q_0(i + 1, j) & q_0(i + 1, j + 1) \\
qu(i, j) & qu(i, j + 1) & 0 & 0 \\
qu(i + 1, j) & qu(i + 1, j + 1) & 0 & 0
\end{bmatrix}.
\]

The optimality condition is again a linear system. For every \( 0 \leq s \leq m, 0 \leq t \leq n \) and \( c \in G \), the equation

\[
\sum_{q \in C} \sum_{i=0}^{m} \sum_{j=0}^{n} G(i, j, q, s, t, c) q_{uv}(i, j) = H(s, t, c)
\]

is obtained from

\[
\frac{\partial E}{\partial c_{uv}(s, t)} = \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{\partial E^{ij}}{\partial c_{uv}(s, t)}.
\]

Each MPD \( c_{uv}(s, t) \) may appear in the energy expression of one (if \( (i, j) \) is a corner), two (if \( (i, j) \) is on the edge) or four \( E^{ij} \)'s. Computing closed formulae for the coefficients \( G(i, j, q, s, t, c) \) and \( H(s, t, c) \) is therefore unnecessarily complicated. A much more reasonable way for computing these coefficients is by going in a loop over all the patches and updating the four coefficients relevant to each patch as we go. The algorithm is:

1. Initialize all \( G(k, l, q, s, t, c) \) and \( H(s, t, c) \) by setting them to zero
2. For \( 1 \leq i \leq m - 1 \) and \( 1 \leq j \leq n - 1 \) do
   1. For \( s = 0, 1 \) and \( t = 0, 1 \) and \( c \in \{ x, y, z \} \) do
      1. \( H(i + s, j + t, c) = H(i + s, j + t, c) + h^{ij}(s, t, c) \)
   2. For \( k = 0, 1 \) and \( l = 0, 1 \), and \( q \in \{ x, y, z \} \) do
      1. \( G(i + k, j + l, q, i + s, j + t, c) = G(i + k, j + l, q, i + s, j + t, c) + g^{ij}(k, l, q, s, t, c) \)
   3. End do
   4. End do
3. End do

The \( h^{ij} \)'s and \( g^{ij} \)'s are taken from the expression of \( \partial E^{ij}/\partial c_{uv}(s, t) \) above. The closed formulae for the coefficients are:

\[
H(i, j, c) = \sum_{s=0}^{1} \sum_{t=0}^{1} h^{i-s,j-t}(s, t, c)
\]

for all \( c \in \{ x, y, z \} \), taking \( h^{ij} = 0 \) if \( i < 0 \) or \( j < 0 \);

\[
G(i, j, q, i+s, j+t, c) = \sum_{k=0}^{1} \sum_{l=0}^{1} g^{i-k,j-l}(k, l, q, s, t, c)
\]

for all \( q, c \in \{ x, y, z \} \) and \( s, t \in \{ 0, 1 \} \) such that \( i+s < m \) and \( j+t < n \), taking \( g^{ij} = 0 \) if \( i < 0 \) or \( j < 0 \); \( G(i, j, q, s, t, c) = 0 \) whenever \( s > i + 1 \) or \( t > j + 1 \).

One additional step is required for our most general case, where the prescribed grid points \((u_i, v_j)\) come from arbitrary knot sequences \( u_0 < u_1 < \cdots < u_m \) and \( v_0 < v_1 < \cdots < v_n \). (So far we have assumed \( u_i = i \) and \( v_j = j \).) We would probably run out of notations (and lose even our most patient reader), if we insisted on filling in all the details. Instead, this last step is only outlined here. The main difference is in the blending functions: \( \alpha_0(u - i) \) is replaced by the (unique) cubic polynomial \( \alpha_0(u) \) which satisfies: \( \alpha_0(u_i) = 1 \) and \( \alpha_0(u_{i+1}) = \alpha_0(u_i) = \alpha_0(u_{i+1}) \).
Fig. 1. The derivative surface for the case with zero twist vectors.

\[ r(u, v) = uv \]

\[ r_{uv}(i, j) = 1, \quad 1 \leq i, j \leq 4, \]

which correspond to the expected surface \( r = uv \).

Zero MPDs \( r_{uv}(i, j) = 0 \) have been known to produce undesired wrinkles. These show up in our case in the plotted partial derivative \( \partial r/\partial v \) (Fig. 1). The same partial derivative for the optimal solution is the plane \( \partial r/\partial u = v \) (Fig. 2). Plotting the surface itself \( r(u, v) \) shows no difference between the two.
Conclusion

In this paper, we have developed a method for automatically selecting optimal values for twist vectors at the grid points in the design of interpolating surfaces. Our approach has led to a general surface design technique based on minimizing an energy functional, which is a general quadratic form in the second partial derivatives of the surface. There are, however, several unanswered questions open for future research. An important area is an evaluation of different terms in the quadratic form and their effects on the shape of the surface. Another question is related to the system of linear equations: If \( \max(m, n) < 4 \), then it seems to be consistently singular, regardless of the choice of quadratic form. (Note that the coefficient matrix of the linear system depends on the blending functions only. It does not depend on the geometric constraints, whose effect is restricted to the free term of the equations.) The theoretical reason for this is not known to the authors at this time. If either \( m \geq 4 \) or \( n \geq 4 \), then the system appears to be non-singular, but we have no proof for it (of course \( \min(m, n) \geq 2 \)).

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