One-to-One Piecewise Linear Mappings Over Triangulations

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Presentation by Elif Tosun

Geometric Modeling

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Outline

- Introduction
- Triangulations
- Convex Combination Mappings
- Proof:
  - Local Injectivity
  - Global Injectivity
- Example Parametrizations

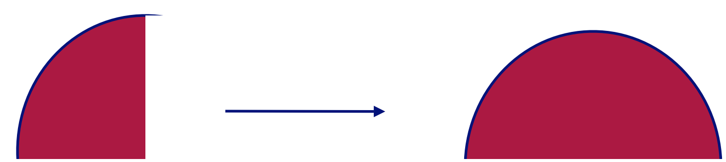
Introduction

- Condition for injectivity of piecewise linear mappings over triangulations
- Applications in
  - Geometric Modeling
  - Computer Graphics
  - Numerical Grid Generation

  } Parametrizations and image morphing
- Discrete analog of a property of harmonic mappings: Radó-Kneser-Choquet Theorem(RKC Thm)

Harmonic Mappings

- Harmonic function:
  - defined on closed region $D \subset \Re^2$
  - satisfies the Laplace Eqn $u_{xx} + u_{yy} = 0$ in the interior of *D*.
- Harmonic mapping-both components are harmonic
- Ex: power functions, Z ® Z^2



- RKC Thm:Given $\phi : D$ ® $\Re^2$ is harmonic and maps the boundary ∂D *homeomorphically* into the convex boundary $\partial\Omega$. Then ϕ is one-to-one.

In this paper:

- A similar property for "convex combination mappings" is established.
- Proof closely follows that of RKC Thm.

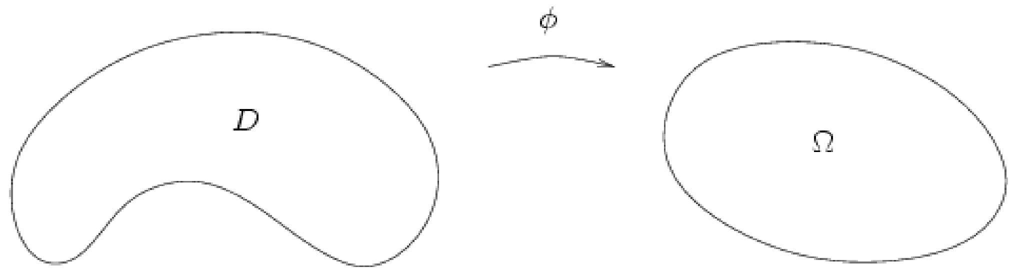


Figure 1. Harmonic map

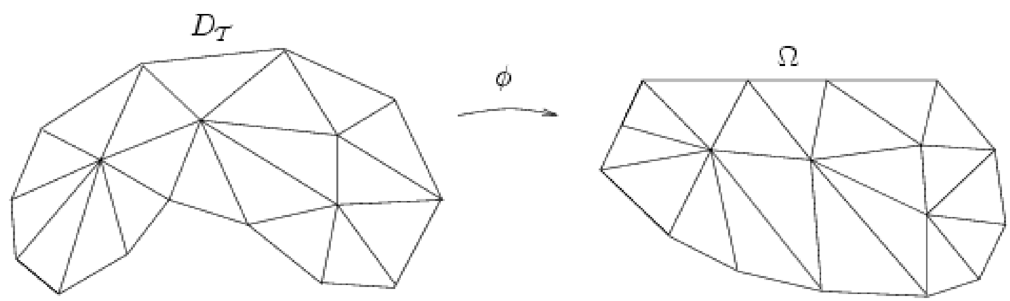
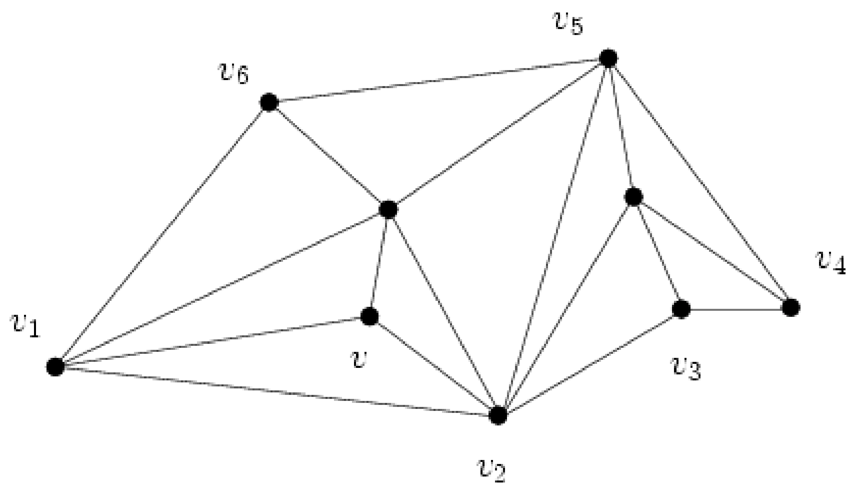


Figure 2. Convex combination map

Triangulations

- Let $\mathcal{T}$ be a finite set of non-degenerate triangles and let $D_\tau = \bigcup_{T \in \tau} T$. $\mathcal{T}$ is a triangulation if
  - The intersection of any pair of triangles is either empty, a common vertex, or a common edge
  - The edges in $\mathcal{T}$ that belong to only one triangle form a simple closed polygon ∂D_τ (=simply connected)
- Dividing Edge: An edge [v, w] of $\mathcal{T}$ that is an interior edge and both v and w are boundary vertices.



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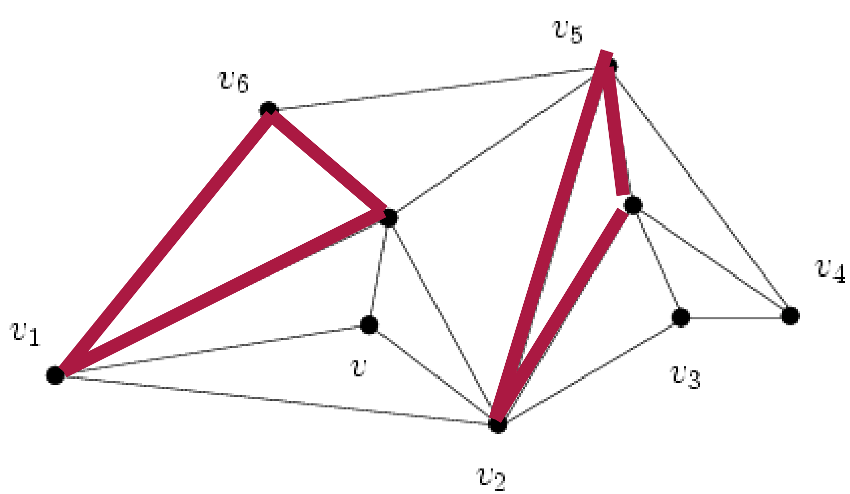

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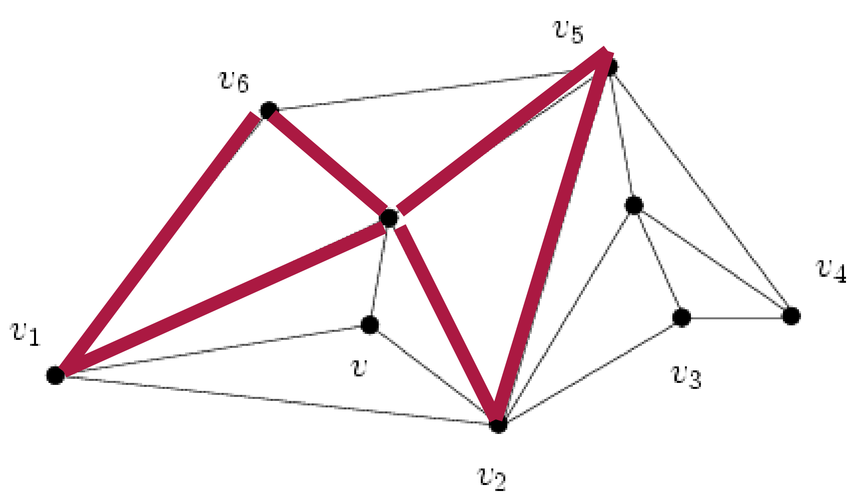

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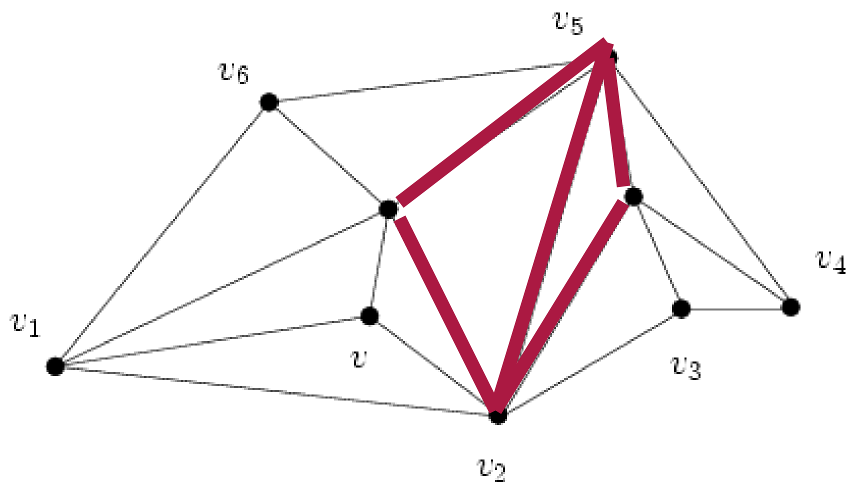

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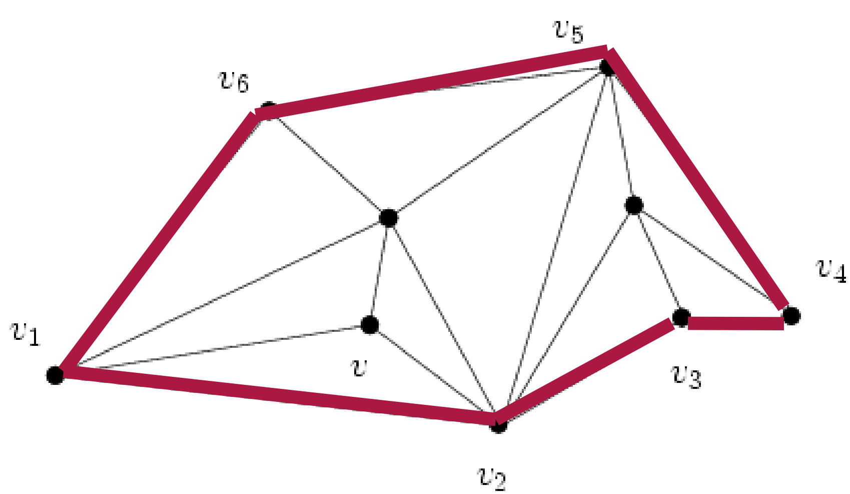

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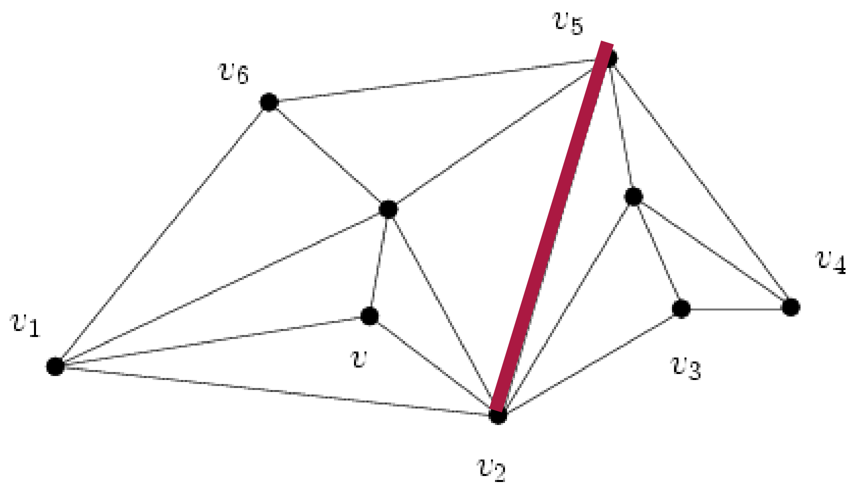
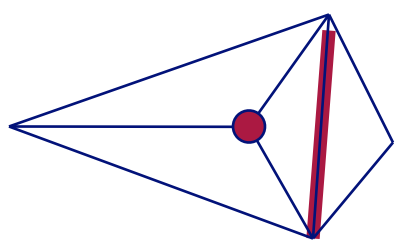
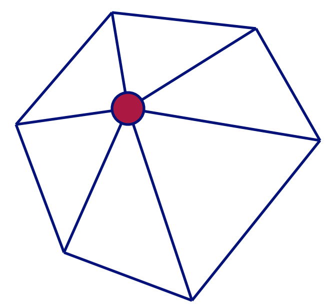


- Lemma:
  - Every interior vertex of $\mathcal{T}$ can be connected to at least 3 boundary vertices by an interior path.
  - If $\mathcal{T}$ contains no dividing edges then v_1 can be connected to every boundary vertex by an interior path. (Proof)

- A triangulation is strongly connected if it contains no dividing edges.

next

•Lemma:

– Every interior vertex of $\mathcal{T}$ can be connected to at least *3* boundary vertices by an interior path.

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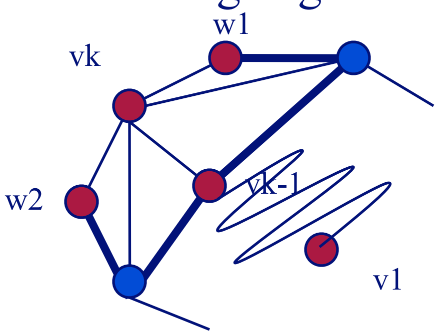
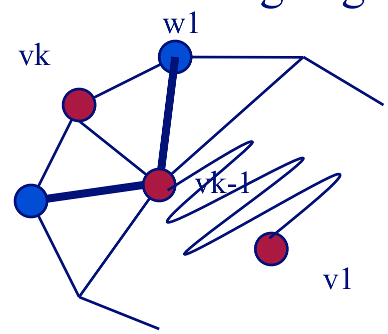
w/ dividing edges



w/out dividing edges



back

Convex Combination Mappings

- Given $\mathcal{T}$, a function $f : D_\tau ® \Re$ is piecewise linear if continuous over $D_\mathcal{T}$ and is linear over each triangle.
- $f : D_\tau ® \Re$ is a piecewise linear function and for every interior vertex v of $\mathcal{T}$, there exist weights $\lambda_{vw} > 0$, for $w \in N_v$ such that $$\sum_{w \in N_v} \lambda_{vw} = 1$$
$$f(v) = \sum_{w \in N_v} \lambda_{vw} f(w)$$
Then, f is a convex combination function.
- $\phi = (u, v)$ is a convex combination mapping if, for every interior vertex v of $\mathcal{T}$, there are positive weights such that $$\phi(v) = \sum_{w \in N_v} \lambda_{vw} \phi(w)$$

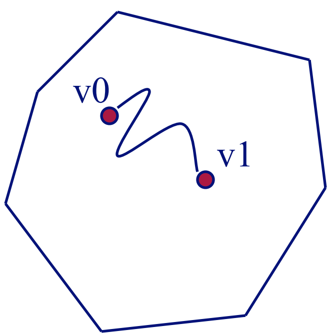
Convex Combination Mappings

- Discrete Maximum Principle: f is a convex combination function over $\mathcal{T}$. For any interior vertex v_0 of $\mathcal{T}$, let V_0 denote the set of all boundary vertices which can be connected to v_0 by an interior path. If $f(v_0) \geq f(v)$ for all $v \in V_0$, then $f(v_0) = f(v)$ for all $v \in V_0$.
- Pf:

W: set of vertices including v_0 which can be connected via interior path

v_1: interior vertex in W at which f attains max=M over all W.

$$M = f(v_1) \geq f(v_0) \geq f(v), \forall v \in V_0$$



Note: maximum of a convex comb. func. is attained at a boundary vertex.

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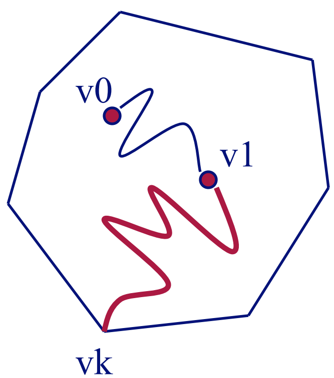


B/c convex combination;

$f(y)=M$ for all neighbors y of v_1.

Then $f(v_k)=M=f(v_0)$.

Note: maximum of a convex comb. func. is attained at a boundary vertex.

Overview

Prove: If $\mathcal{T}$ is a strongly connected triangulation and ϕ is a convex combination mapping that maps ∂D_τ to $\partial\Omega$ of convex Ω. Then ϕ is one-to-one.

Step1:

Local Injectivity: $\phi|_Q$ is one-to-one for every quadrilateral Q in $\mathcal{T}$.

Step2

Global Injectivity

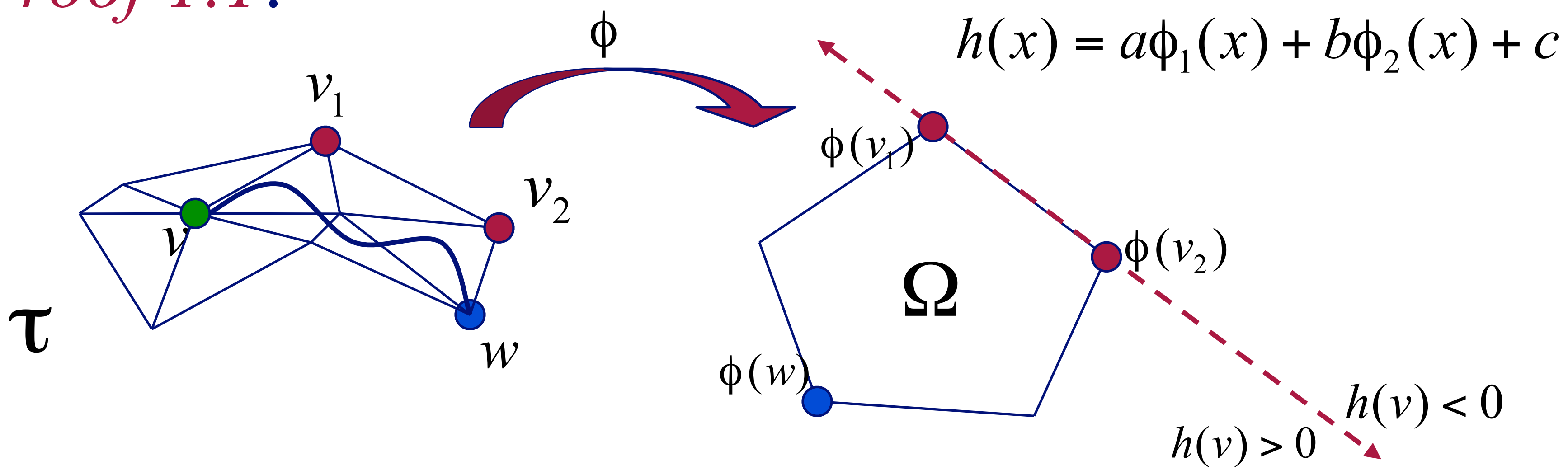
Local Injectivity

ϕ *is locally one-to-one in the sense that* $\phi|_Q$ *is one-to-one for every quadrilateral* Q *in* $\mathcal{T}$.

Proof:

Lemma 1.1: For every interior vertex v of $\mathcal{T}$, the point $\phi(v)$ lies in the interior of Ω.

Proof 1.1:



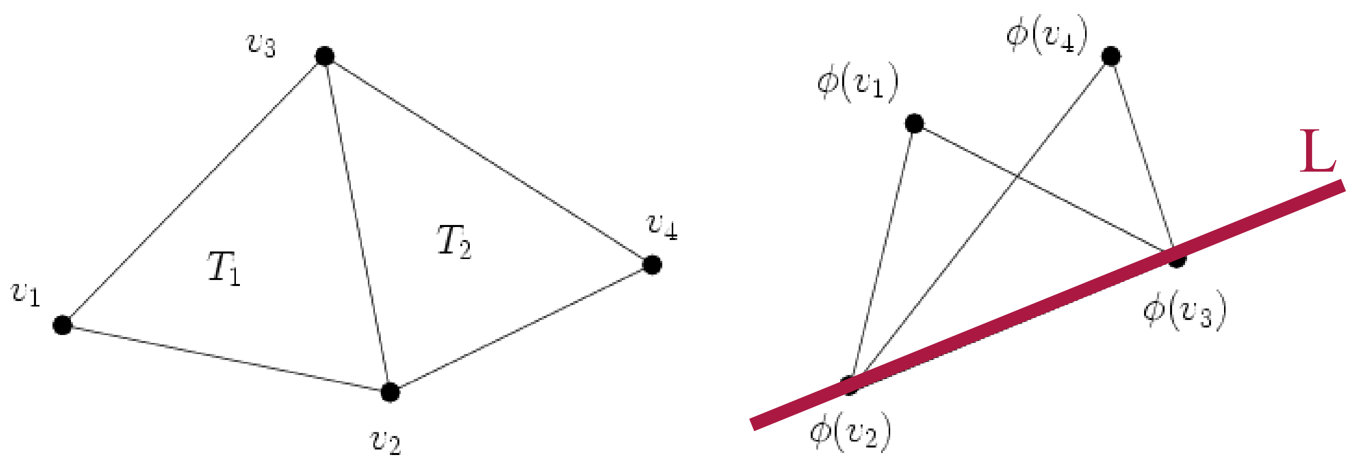
Local Injectivity

Note: From Lemma 1.1:

- Every triangle T containing a boundary edge in $\mathcal{T}$, is mapped to a non degenerate triangle $\phi(T)$.

Lemma 1.2: If $T_1 \cup T_2$ is a quadrilateral in $\mathcal{T}$ and if $\phi|_{T1}$ is one-to-one then $\phi|_{T1 \cup T2}$ is one-to-one

Proof 1.2:



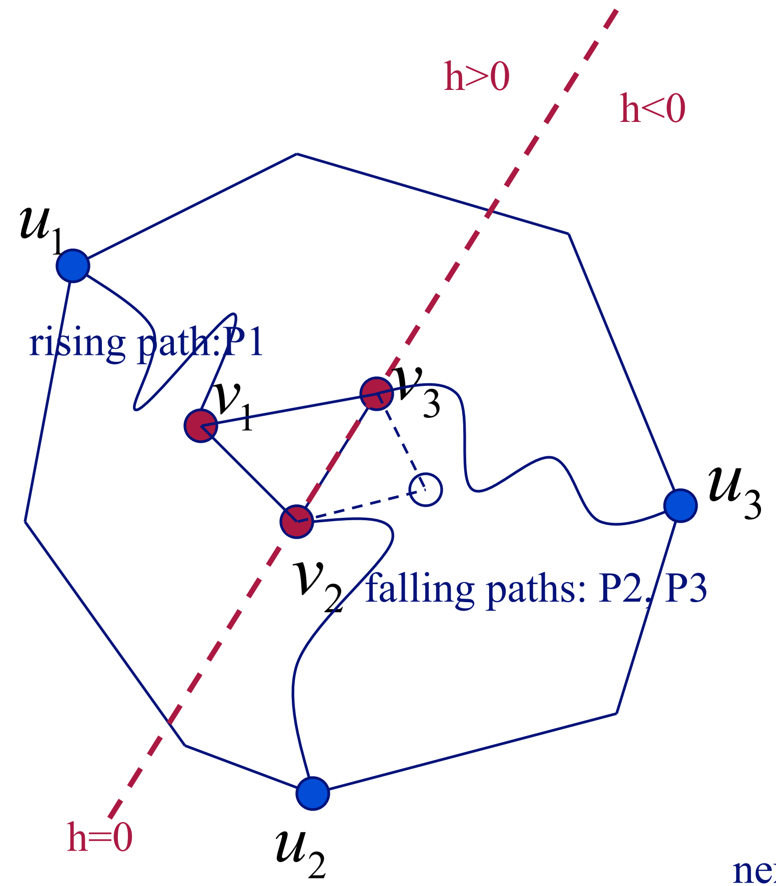
Local Injectivity

Use eqn of L to come up with h: $h(x) = a\phi_1(x) + b\phi_2(x) + c$

Then *h* is a convex combination function satisfying:

$h(v_2)=h(v_3)=0$ and $h(v_1)>0$ and $h(v_4)>=0$.

Note:
-P1 is distinct from P2 and P3.
-P2 is also distinct from P3.(proof)



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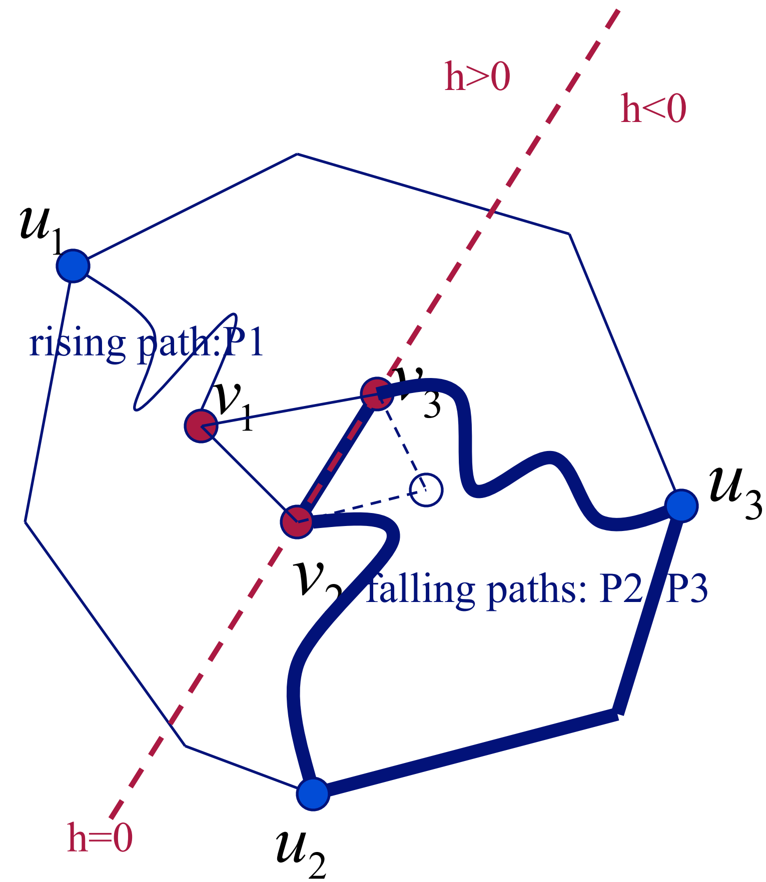
Now let Q be the closed path P2, P3, and edge (v2, v3).



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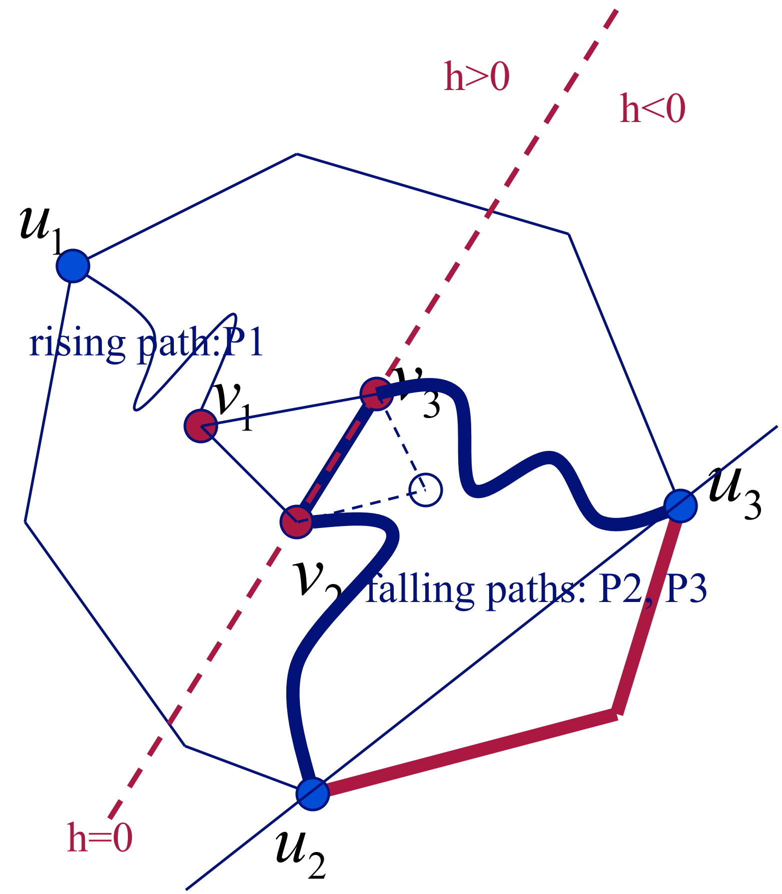
-P2 is also distinct from P3.(proof)

Now let Q be the closed path P2, P3, and edge (v2, v3).

Due to convexity, and since h(u2)<0, h (u3)<=0, we have h(v)<0 for all boundary vertices v of T in Q other than u2 and u3.

Then h(v)<0 for every vertex v in Q other than v2 and v3.

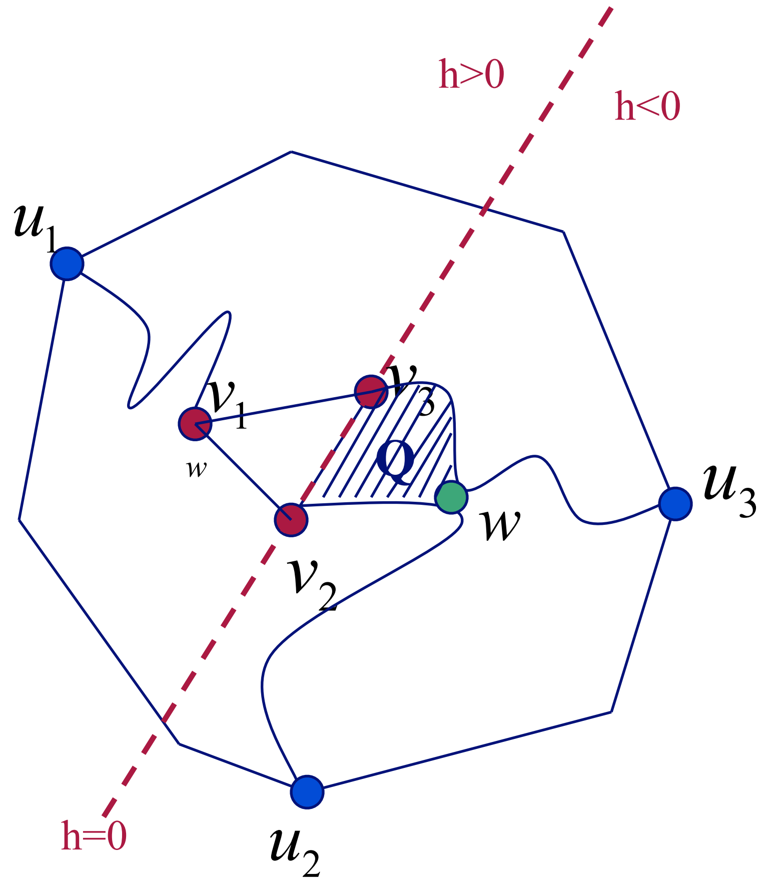
Since v4 in enclosed by Q, and h(v4)>=0; contradiction.



Q cannot enclose v1, for then Q would have to cross P1.

Then Q would have to enclose v4.

But cannot since we said h(v4)>=0, and the discrete max principle over Q fails (b.c v4 must be connected to at least one other vertex than v2 and v3 and due to falling path h (v)<0).

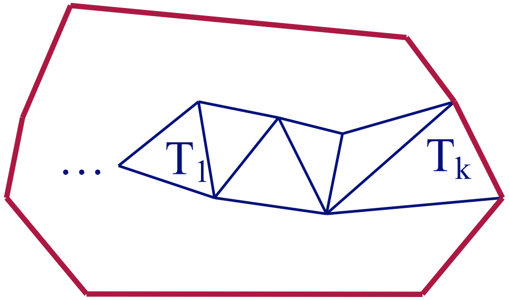


Local Injectivity

Due to connectedness of area of $\mathcal{T}$:

- Any 2 triangles in $\mathcal{T}$ can be connected by a path of triangles where each consecutive pair shares an edge.
- So any T_1 in $\mathcal{T}$ can be connected to some T_k that has a boundary edge. By Lemma 1.1 , T_k is non-degenerate. Then by Lemma 1.2, the T_{k-1} is non-degenerate, and so on. Then ϕ is one-to-one on every triangle, and therefore is one-to-one on every quadrilateral.

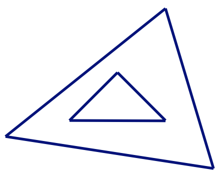
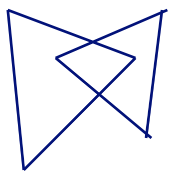
Q.E.D



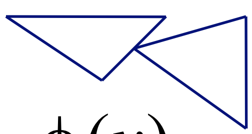
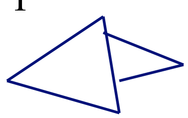
Step2 : Global Injectivity

Overview:

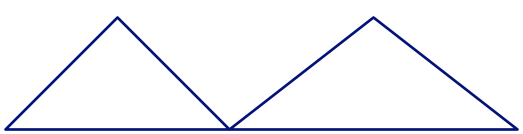
- For any two distinct triangles T&S in $\mathcal{T}$, $\phi(T)$ and $\phi(S)$ have disjoint interiors.

- If T_1 and T_2 in $\mathcal{T}$ don't share an edge, then $\phi(T_1) \cap \phi(T_2)$ is either empty or a point $\phi(v_1) = \phi(v_2)$, for $v_1 \in T_1$ and $v_2 \in T_2$.

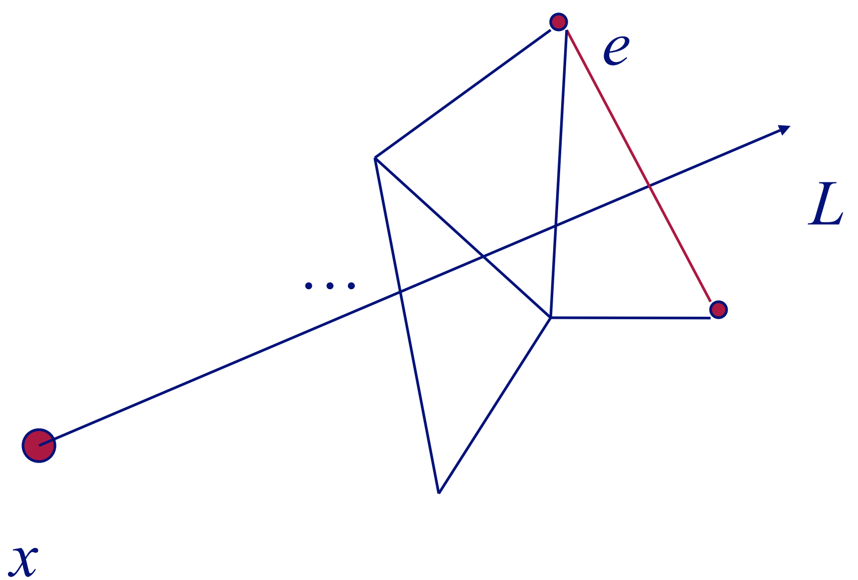
- If $T_1 \cap T_2 = v$ then $\phi(T_1) \cap \phi(T_2) = \phi(v)$



- If $T_1 \cap T_2 = \phi$ in $\mathcal{T}$, then $\phi(T_1) \cap \phi(T_2)$ is also empty.

Global Injectivity

- *Lemma 2.1*: For any two distinct triangles T and S in $\mathcal{T}$, $\phi(T)$ and $\phi(S)$ have disjoint interiors.
- *Pf*: Assume there exists an x s.t. x belongs to interiors of both $\phi(T)$ and $\phi(S)$.



Path: T_1-> T_k

Path: S_1-> S_l

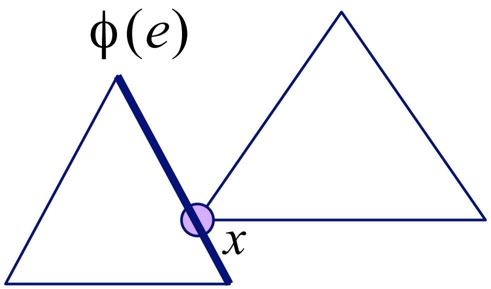
2 cases:

$l = k$, $T=S$!

$k>l$, $T_{k-l+1}=S$!

Global Injectivity

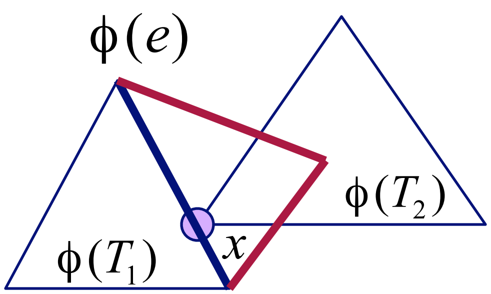
- *Lemma 2.2*: If T_1 and T_2 in $\mathcal{T}$ don't share an edge, then $\phi(T_1) \cap \phi(T_2)$ is either empty or a point $\phi(v_1) = \phi(v_2)$, for $v_1 \in T_1$ and $v_2 \in T_2$.
- *Pf*: Suppose the intersection is x. *Note e* cannot be boundary edge.



Global Injectivity

- *Lemma 2.2*: If T_1 and T_2 in $\mathcal{T}$ don't share an edge, then $\phi(T_1) \cap \phi(T_2)$ is either empty or a point $\phi(v_1) = \phi(v_2)$, for $v_1 \in T_1$ and $v_2 \in T_2$.
- *Pf*: Suppose the intersection is x. *Note* e cannot be boundary edge.

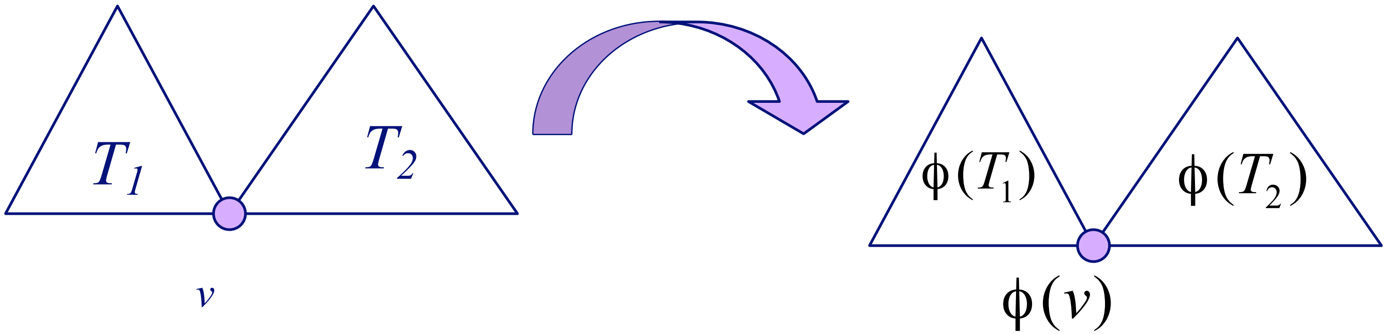
e must be shared between T1 and some other triangle T3 != T2.



Interior intersection -> contradict prev lemma

Global Injectivity

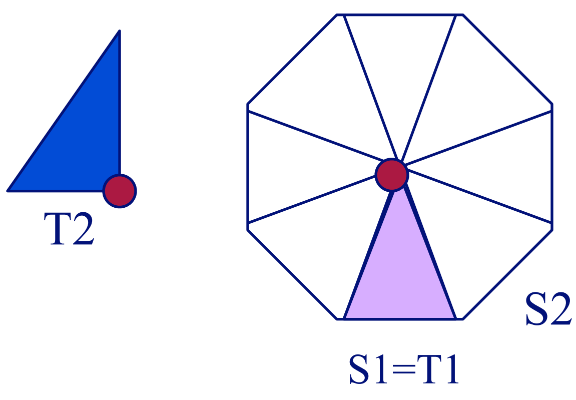
- *Lemma 2.3*: If $T_1 \cap T_2 = v$ then $\phi(T_1) \cap \phi(T_2) = \phi(v)$
- *Pf*:

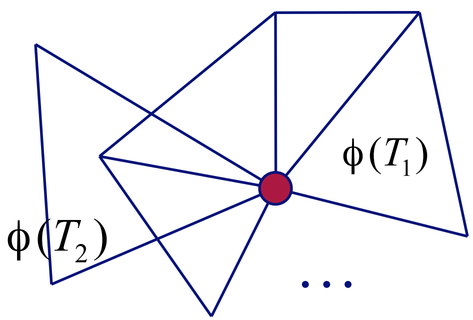


Because of lemma 2.1 and 2.2 intersection is either empty or a common vertex. Since v is common, then intersection is $\phi(v)$

Global Injectivity

- *Lemma 2.4*: If $T_1 \cap T_2 = \phi$ in $\mathcal{T}$, then $\phi(T_1) \cap \phi(T_2)$ is also empty
- *Pf*: Suppose not, and then they intersect at a vertex. (b/c of Lemma 2.2) It has to be an interior vertex.





Proof of main theorem

If $\mathcal{T}$ is a strongly connected triangulation and ϕ is a convex combination mapping that maps ∂D_τ to $\partial\Omega$ of convex Ω . Then ϕ is one-to-one.

- Let x_1 and x_2 be distinct points in $D\tau$.
  - If they belong to a common triangle T, then from local injectivity, we know $\phi(x_1) \neq \phi(x_2)$.
  - Then say x_1 belongs to $T1$, and x_2 belongs to $T2$. But by the global injectivity we know that ϕ is one-to-one in $T_1 \cup T_2$. Therefore: $\phi(x_1) \neq \phi(x_2)$.

QED

Example parametrizations

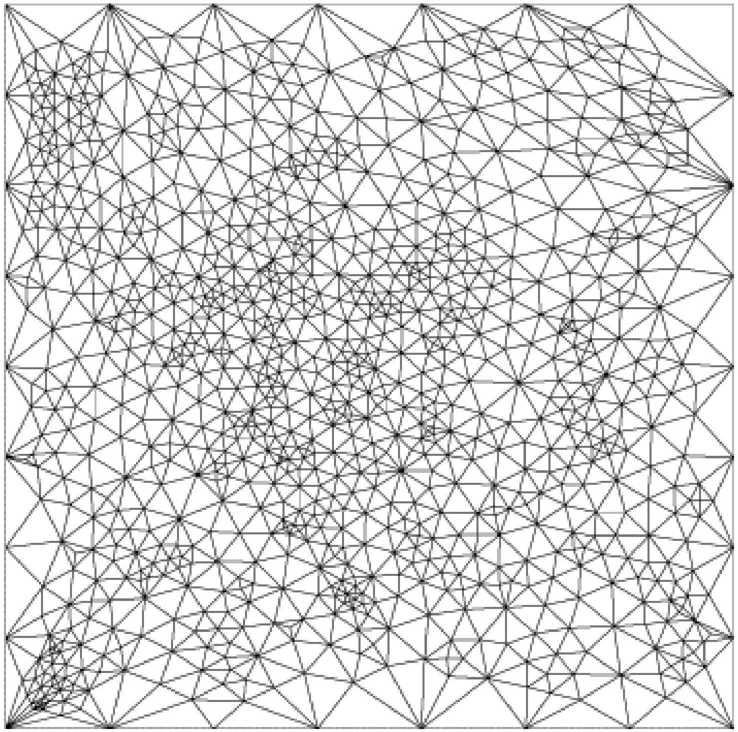


Fig. 11. Uniform parametrization.

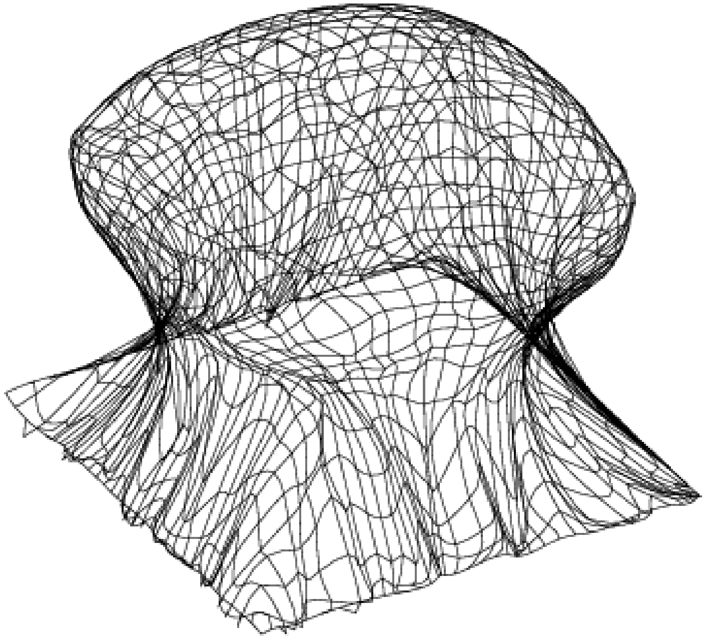


Fig. 12. Surface approximation.

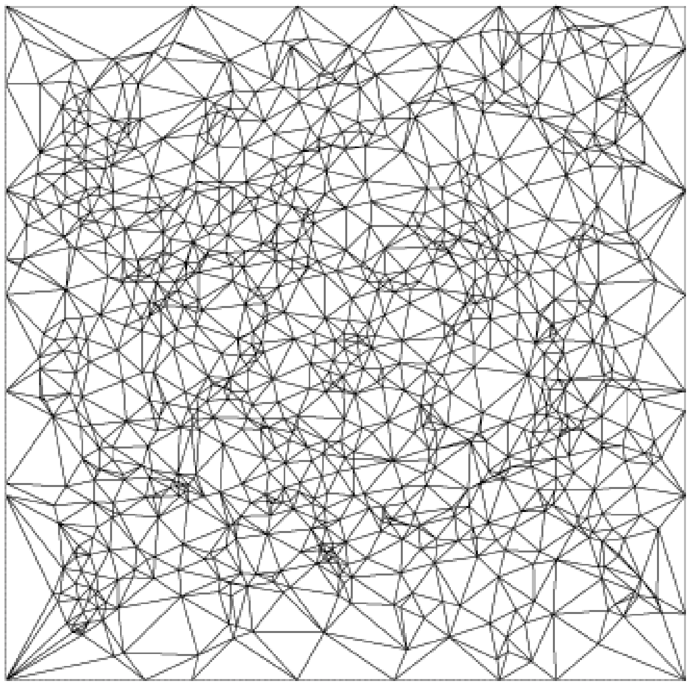


Fig. 13. Weighted least squares parametrization.

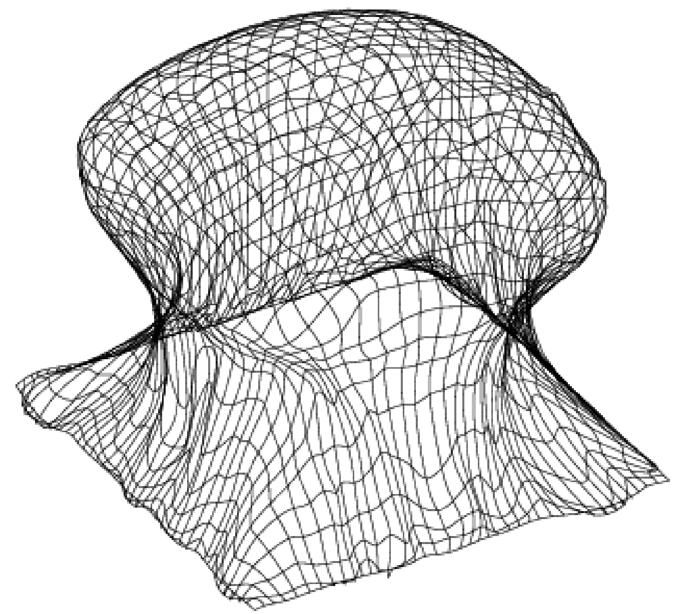


Fig. 14. Surface approximation.

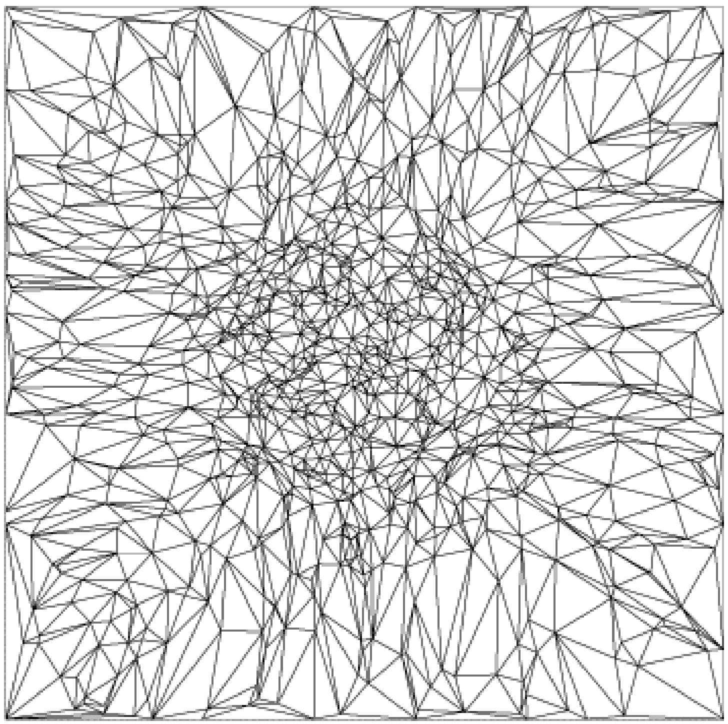


Fig. 15. Shape-preserving parametrization.

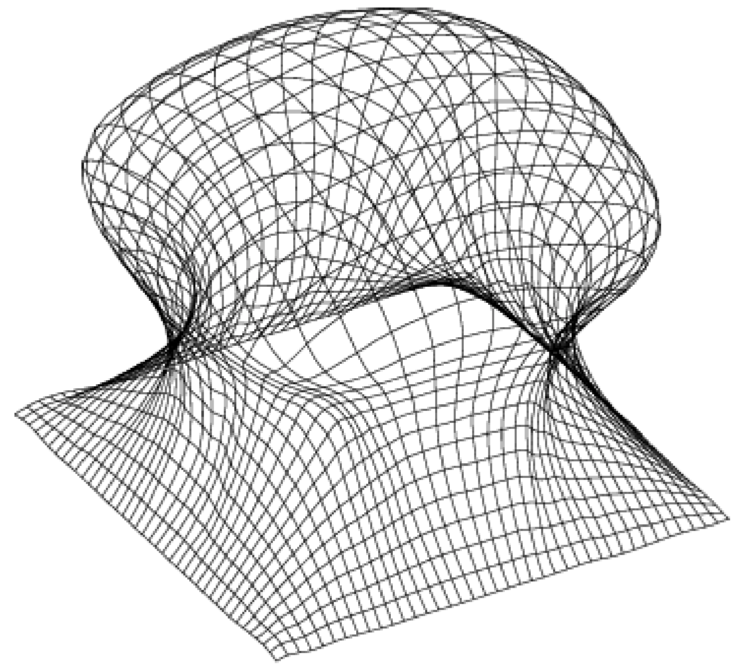


Fig. 16. Surface approximation.

Questions