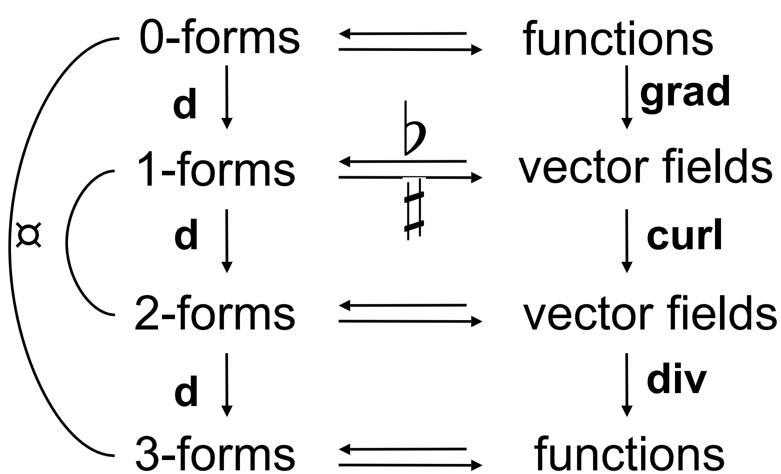
Discrete Exterior Calculus

Notes from
Anil Hirani’s PhD Thesis

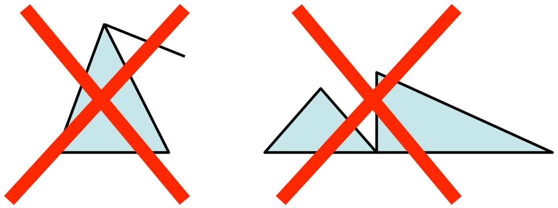
Idea

- Discrete counterpart of smooth exterior calculus.
- Novelty: Simultaneous presence of forms and vector fields

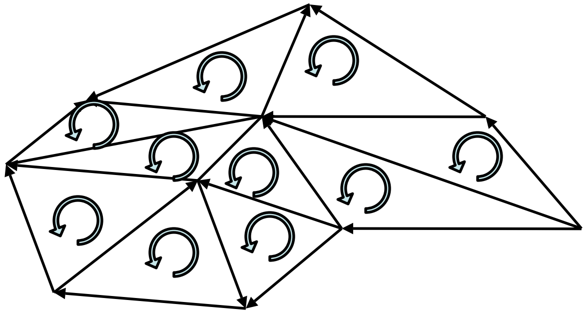


Basics

- Simplicial complex
  - Collection of σ^0 (vertices), σ^1 (edges), σ^2 (triangles)
- Manifold-like
  - Like a valid triangulation.

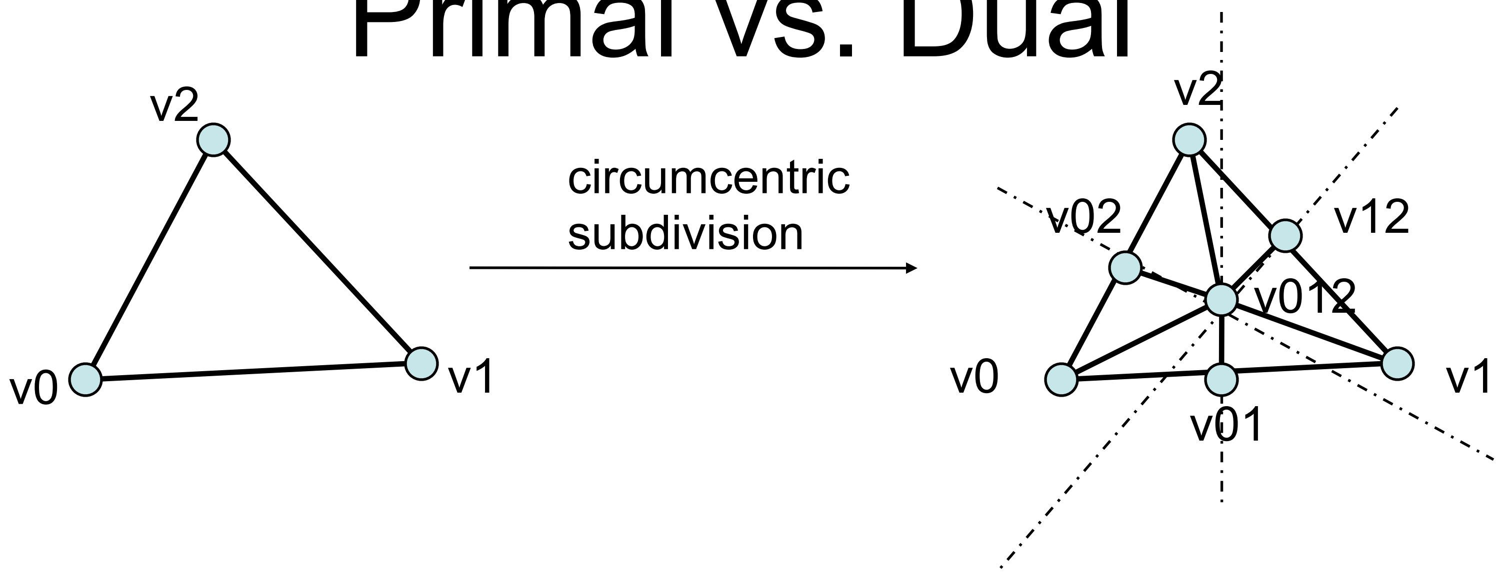


- Oriented



"Primal"

Primal vs. Dual



σ^0 : v0,v1,v2

σ^1 : [0,1],[1,2],[2,0]

σ^2 : [0,1,2]

σ^0 : v0,v1,v2,v01,v12,v02,v012

σ^1 : [0,01],[01,1],[1,12],[12,2], [2,02],[02,0],[0,012],[01,012], [1,012],[12,012],[2,012],[02,012]

σ^2 : [0,01,012],[01,012,1], [012,1,12],[012,12,2], [012,2,02], [0,012,02]

"well-centered"

Primal vs. Dual

primal

$\phi_{(\sigma, D(\sigma))}$ &

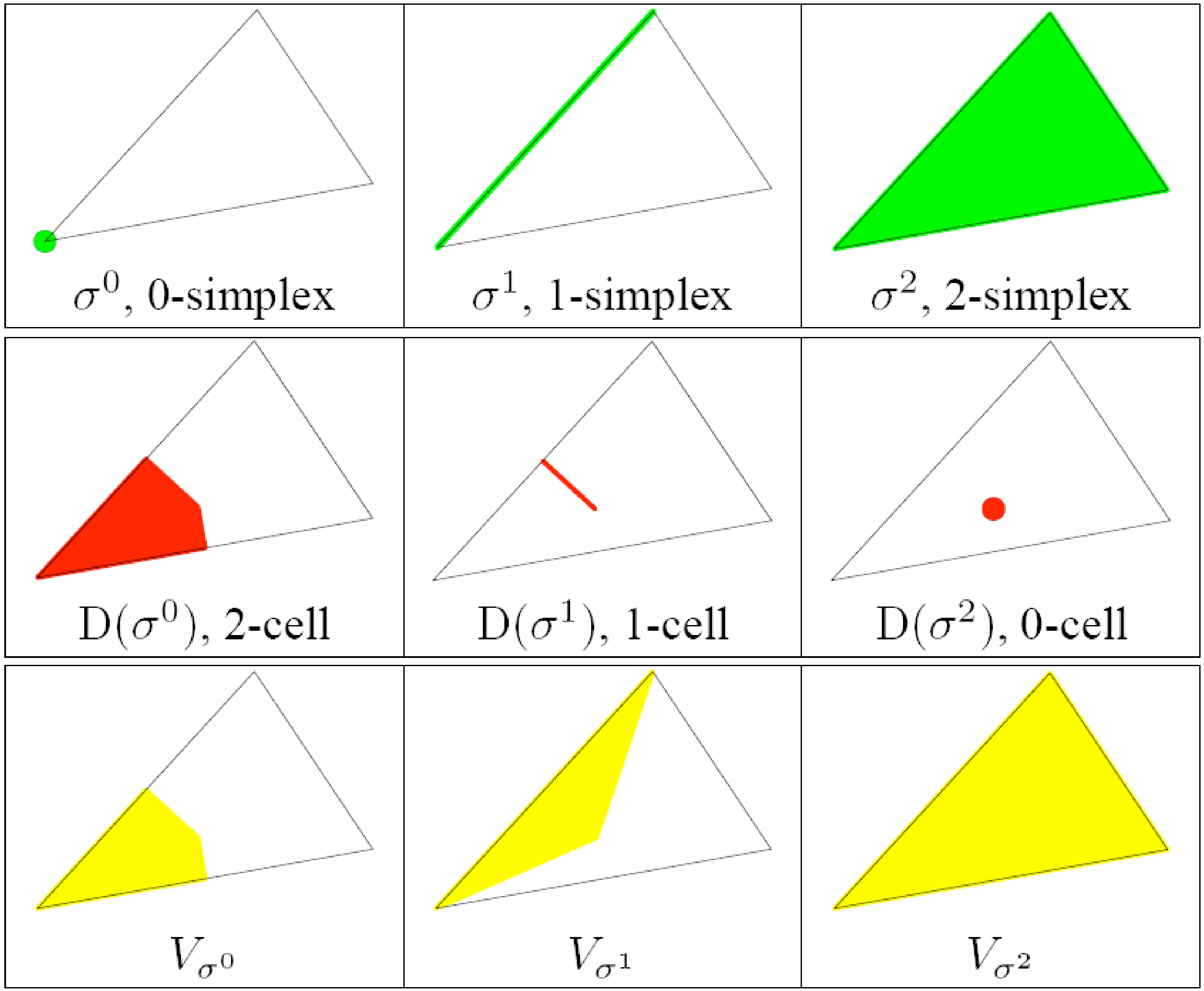
$\phi_{(D(\sigma), \sigma)}$:

Interpolation functions

dual

•(p, n-p) relation

support volume



Differential forms

- Smooth:

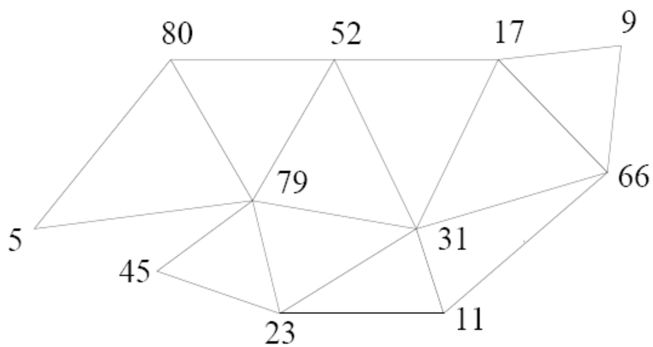
  "Integration of a p-form on a p-manifold"

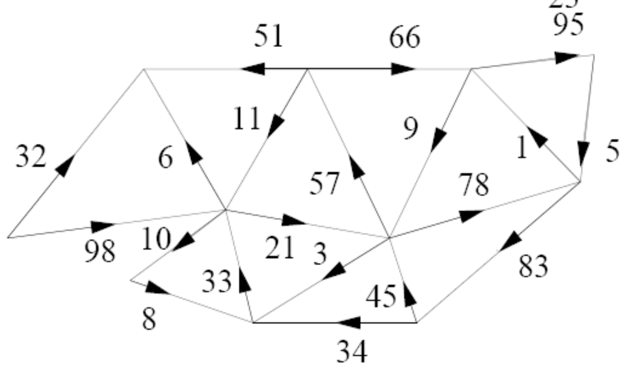
- Discrete:

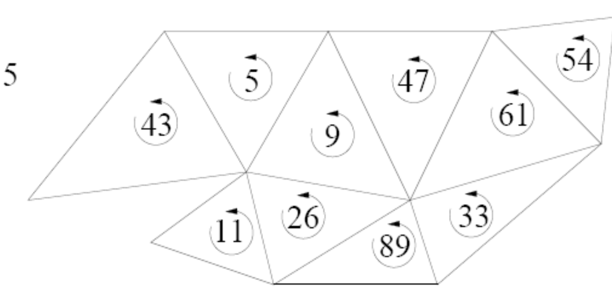
  "Evaluation of a discrete p-form on a p-chain"

Map from p-chains to R (cochains)

Like an array of p-simplices of a complex K







$$\Omega_d^p(K)$$

$$\Omega_d^p(\star K)$$

Operators: Boundary Operator

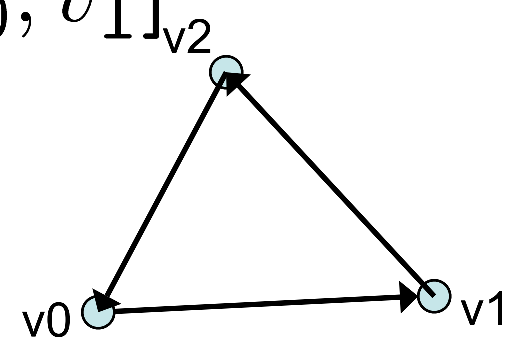
Boundary operator on a simplex

$$\sigma^p = [v_0, v_1, ...v_p]$$

$$\partial_p \sigma^p = \sum_{i=0}^{p} (-1)^i [v_0, ..., \widehat{v_i}, ..., v_p]$$

e.g. p = 2, a triangle

$$\partial_2 \sigma^2 = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

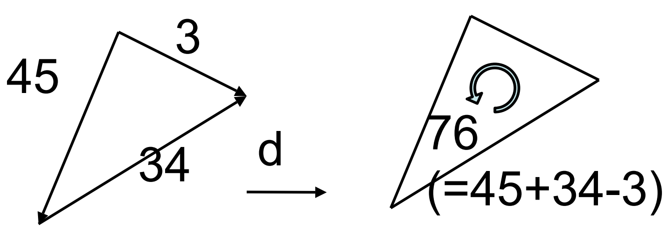


Operators: Disc. Ext. Derivative

- Coboundary operator = Disc. Ext.Derivative

$$< \delta^p \alpha^p, c_{p+1} > = < \alpha^p, \partial_{p+1} c_{p+1} >$$

- Disc. Ext. derivative d = δ: $\Omega_d^p(K) \rightarrow \Omega_d^{p+1}(K)$



"map from a p-form (evaluated on the edge) to a (p+1)-form (evaluated on the triangle) "

- Stokes' Theorem: $< d\alpha, c > = < \alpha, \partial c >$

Operators: Hodge Star $*$

- Smooth case: $*$ is an isomorphism between the space of p-forms and (n-p)-forms
- Discrete case:

Reminder: dual of a p-simplex is an (n-p)-cell

$$* : \Omega_d^p(K) \rightarrow \Omega_d^{(n-p)}(\star K)$$

For a p-simplex σ^p and a discrete p-form α^p:

$$\frac{1}{|\star\sigma^p|} < *\alpha, \star\sigma^p > := \frac{1}{\sigma^p} < \alpha, \sigma^p >$$

Operators: Codifferential

- “inverse” of discrete exterior derivative

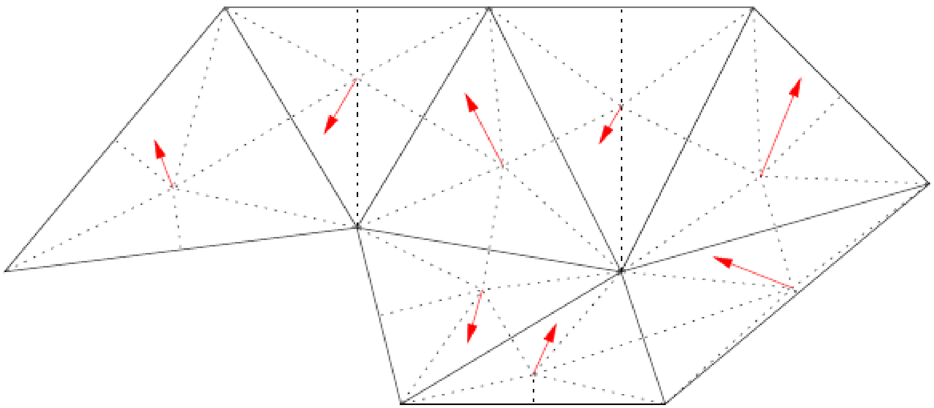
$$\delta : \Omega_d^{p+1}(K) \rightarrow \Omega_d^p(K)$$

- β : (p+1)-discrete form

$$\delta\beta = (-1)^{np+1} * d * \beta$$

- For a 0-form: $\delta(\Omega_d^0(K)) = 0$

Discrete Vector Fields

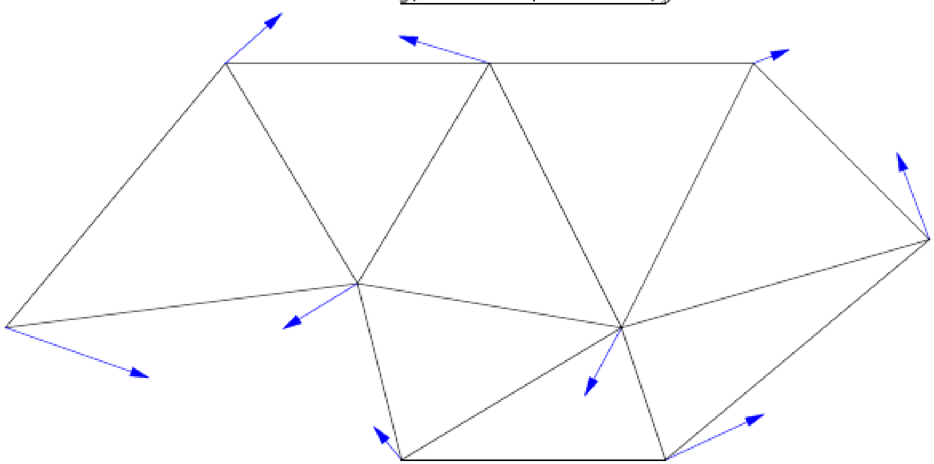


Can interpolate to:

dual → D : $\sum X\phi_{(D(\sigma),\, D(\sigma))}$:

Or

P : $\sum X\phi_{(D(\sigma),\, \sigma)}$



primal → D : $\sum X\phi_{(\sigma,\, D(\sigma))}$:

Or

P : $\sum X\phi_{(\sigma,\, \sigma)}$

+

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4 types of vector fields!

Operators: Sharp $\sharp$ & Flat $\flat$

Smooth case:

Given a manifold with a metric <<,>>(Riemanninan),

X: vector field, v: tangent vector, α: 1-form

Sharp : 1 forms → vector fields

$$<< \alpha^{\sharp}, v >> = \alpha(v)$$

Flat : vector fields → 1 forms

$$<< X, v >> = X^{\flat}(v)$$

Sharp and Flat are inverses!

Operators: Sharp & Flat

- Discrete flats:
  - 2 types of vector fields
  - 2 interpolation methods
  - × 2 types of forms

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  8 different flat operators : $\flat_{ppp}, \flat_{pdp}, \flat_{ppd}, \flat_{pdd},$ $\flat_{dpp}, \flat_{ddp}, \flat_{dpd}, \flat_{ddd}$
- Discrete sharps:
  – Ad-hoc definitions with no interpolation

  2 vecfields x 2 forms = 4 sharp operators

  $\sharp_{pp}, \sharp_{pd}, \sharp_{dp}, \sharp_{dd}$
- Inverse problem!

Operators: Discrete Flat

- Y smooth vector field, r a smooth curve :

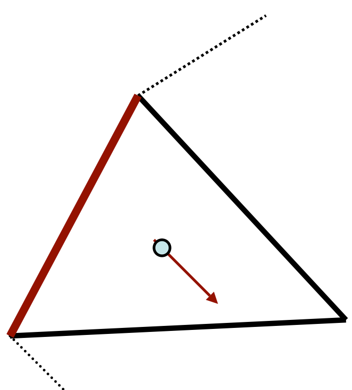
$$\int_r Y^\flat = \int_0^L Y(r(s)) \cdot \dot{r}(s) ds$$

- X: discrete vector field, r: edge (pw-sm. curve)

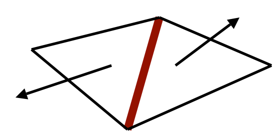
$$< X^\sharp, r > := \int_r (\sum X\phi)^\flat$$

$$= \underbrace{\sum X\phi}_{\text{Vec field at an edge}} \cdot \underbrace{\vec{r}}_{\text{Vec along } r \text{ with length} = |r|}$$

$\flat_{dpp}$



Problem:



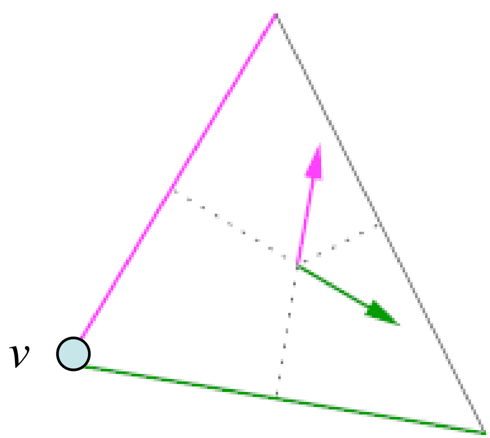
Solution: weighted averages (sup. vol.)

Operations: Discrete Sharp

$\sharp_{pd}$

- Given f: 0-form, v: any vertex of a triangle

$$< (df)^{\sharp}, \star\sigma^2 > := \sum_{\sigma^0 \prec \sigma^2} (f(\sigma^0) - f(v)) \nabla \phi_{\sigma,\sigma}$$



Divergence

$$div(X) = *d * X^\flat$$

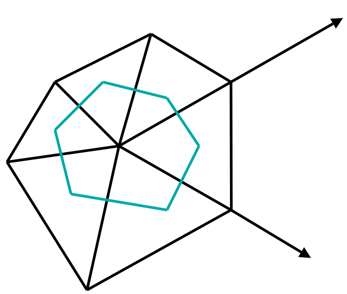
Smooth divergence thm:

Integral of div(X) over a region   Flux of X through boundary

$$\int\int\int div(X) dV = \int\int X \cdot n dS$$

Discrete div thm on dual cell

$$|\star\sigma^0| \ \langle \mathrm{div}(X), \sigma^0 \rangle = \sum_{\sigma^1 \succ \sigma^0} \ \sum_{\sigma^n \succ \sigma^1} |\star\sigma^1 \cap \sigma^n| \ \left(X(\star\sigma^n) \cdot \frac{\vec{\sigma}^1}{|\sigma^1|} \right)$$



Divergence

$$div(X) = *d*X^\flat$$

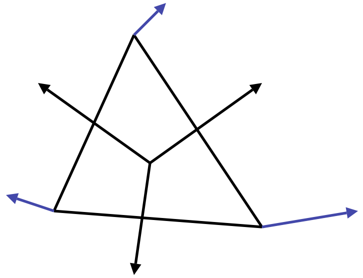
Smooth divergence thm:

Integral of div(X) over a region | Flux of X through boundary

$$\int\int\int div(X)dV = \int\int X \cdot ndS$$

Discrete div thm on a primal 2-simplex

$$|\sigma^n| \ \ \langle \mathrm{div}(X), \star\sigma^n \rangle = \sum_{\sigma^{n-1} \prec \sigma^n} |\sigma^{n-1}| \left(\sum_{\sigma^0 \prec \sigma^{n-1}} X(\sigma^0) \right) \cdot \frac{\star\sigma^{n-1}}{|\star\sigma^{n-1}|}$$



Gradient

- Smooth: $\nabla f = (df)^\sharp$

- Discrete PD-gradient:

f : primal 0-form , v any vertex

$$< (df)^\sharp, \star\sigma^2 > := \sum_{\sigma^0 \prec \sigma^2} (f(\sigma^0) - f(v)) \nabla \phi_{\sigma,\sigma}$$

Curl

- 2D: $*dX^\flat$

$$< curl(X), \sigma^0 >= \frac{1}{|\star\sigma^0|} \sum_{\sigma^2 \succ \sigma^0} X(\star\sigma^2) \cdot \sigma^1_{outer}$$

- Polthier/Preuss '02

$$< curl(X), \sigma^0 >= \frac{1}{2} \sum_{\sigma^2 \succ \sigma^0} X(\star\sigma^2) \cdot \sigma^1_{outer}$$

- 3D: $(*dX^\flat)^\sharp$

Laplace-Beltrami

- Smooth : $\nabla^2 = div \cdot curl = \delta d$
  (special case of Laplace-DeRham : $\triangle = d\delta + \delta d$)

- Discrete:

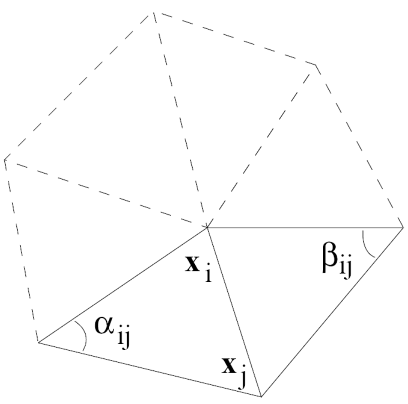
$$< \triangle f, \sigma^0 > = < (d\delta + \delta d)(f), \sigma^0 >$$
$$= < d\delta f + \delta df, \sigma^0 >$$
$$= - < *d*df, \sigma^0 >$$

Laplace-Beltrami

$$< \triangle f, \sigma^0 > = - < *d * df, \sigma^0 >$$

...

$$\langle \Delta f, \sigma^0 \rangle = \frac{1}{|\star\sigma^0|} \sum_{\sigma^1=[\sigma^0,v]} \frac{|\star\sigma^1|}{|\sigma^1|} (f(v) - f(\sigma^0))$$



Meyer et al.02:

$$\Delta f(\mathbf{x}_i) = \frac{1}{2\mathcal{A}} \sum_{j \in N_1(i)} (\cot\ \alpha_{ij} + \cot\ \beta_{ij})\ (f(\mathbf{x}_i) - f(\mathbf{x}_j))$$