Detecting regions with undesirable curvature

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Abstract. The use of Bézier curves or Bézier surfaces as 'faired' interpolation (approximation) curves or surfaces occasionally yields inflection points or changes of sign of the Gaussian curvature which are undesirable in some applications, notably the fairing of ship lines (surfaces) or the fairing of car body surfaces. In the following investigation we use a polarity to detect such undesirable points, then these points can be removed by the use of the well known Bézier techniques.

Keywords. Curvature of Bézier curves, surfaces, polarity.

Introduction

If we approximate (or interpolate) free form curves or surfaces by the aid of Bézier-curves (or B-spline curves respectively) or Bézier surfaces (or B-spline surfaces respectively), the approximating curves may have undesirable inflection points or the approximating surfaces may get regions with undesirable curvature. As an example in a patch expected to be convex (Gaussian curvature $> 0$) a region with negative Gaussian curvature may occur or in a patch expected to have negative Gaussian curvature a region has positive Gaussian curvature. Therefore the following problems arise

- to find a method to recognize an undesirable part on a curve or surface;
- if such an undesirable region is localized to develop a method for removing this undesirable region i. e. for smoothing the curves or surfaces.

[Renz '82] has presented a new approach for smoothing digitized point data and curves respectively. The smoothness of the data is made visible by displaying the first and the second differences. Local corrections are performed on the first and second differences curves. The improved original data points are then computed by integration. [Klass '80] made deviations of a surface visible by irregularities in the reflection lines of a family of straight light lines. To correct these deviations a procedure was developed to find necessary changes of the surface for a given variation of the reflection lines.

The goal of the present paper is to develop a uniform criterion for curves and surfaces, which permits recognition of inflection points of a curve or changes of the sign of the Gaussian curvature of a surface interactively. This criterion leads to a method for removing these undesirable regions. For this purpose we will use a polar image $P$ of a curve or a surface: A point of a curve with vanishing curvature or a curve of a surface with vanishing Gaussian curvature leads to singularities of the polar image (cusps, edges of regression). These singularities can be seen easily on a graphic display if the origin of the polarity is transformed nearby these undesirable regions. The removal of these singularities can be interpreted mathematically by the
concept of the dual Bézier-curves and dual Bézier surfaces introduced by the author [Hoschek '83a].

1. Polarity

We start from a plane curve \( X \) with the parametric representation
\[
X(t) = (x(t), y(t)).
\]
Further we will suppose that the curve does not pass through the origin. For our purpose we will use a polarity with respect to the complex unit circle. A polarity is a linear transformation that maps a point onto a line or vice versa. Therefore the image of a point \( X \) with the parameter \( t = t_0 \) under the polarity with respect to the complex unit circle is the line (polar) [Coxeter '49]
\[
\xi x(t_0) + \eta y(t_0) + 1 = 0.
\]
If the parameter \( t \) runs through the whole range of definition the resulting family of straight lines envelopes the polar curve \( P \) of the curve \( X \). The parameter representation of the polar curve can be obtained by differentiation of (1) and elimination as
\[
\begin{align*}
\xi &= \frac{-\dot{y}}{x\dot{y} - \dot{x}y}, \\
\eta &= \frac{\dot{x}}{x\dot{y} - \dot{x}y},
\end{align*}
\]
with \( x \neq 0, y \neq 0 \) for it is assumed, that the curve does not pass through the origin.

Now we consider a surface in 3-space with the regular parametric representation
\[
X(u, v) = (x(u, v), y(u, v), z(u, v)).
\]
The image of a point \( X(u_0, v_0) \) of the surface under the polarity with respect to the complex unit ball is the plane \( \epsilon \) (polar plane)
\[
\xi x(u,v) + \eta y(u,v) + \zeta z(u,v) + 1 = 0.
\]
If the parameters \( u, v \) run through the whole range of definition the resulting family of the polar planes envelopes the polar surface \( P = P(\xi, \eta, \zeta) \) of the surface \( X \). The parametric representation can be obtained by partial differentiation with respect to \( u \) and \( v \) as
\[
\begin{align*}
\xi &= \begin{vmatrix}
-1 & y(u,v) & z(u,v) \\
0 & y_u & z_u \\
0 & y_v & z_v
\end{vmatrix}, \\
\eta &= \begin{vmatrix}
x(u,v) & -1 & z(u,v) \\
x_u & 0 & z_u \\
x_v & 0 & z_v
\end{vmatrix},
\end{align*}
\]
with \( \Delta = \begin{vmatrix}
x(u,v) & y(u,v) & z(u,v) \\
x_u & y_u & z_u \\
x_v & y_v & z_v
\end{vmatrix} \) (4)

The polar mapping with respect to the complex unit circle or the complex unit ball can be interpreted geometrically as 'mapping with reciprocal radii'. That means: If we denote the distance of the point \( X \) from the origin with \( d = OX \) the polar line (polar plane) has the distance
\[
d_p = \frac{1}{OX} \text{ from the origin} \quad \text{(see Fig. 1).}
\]
2. Smoothing of plane Bézier-curves

We now consider the polar curve $P$ of a given plane curve $X$. The following theorem can be proved [Hoschek '83b]:

**Theorem 1.** If the plane curve $X(t)$ has an inflection point at $t = t_0$, the polar curve $P(t)$ has a singularity (cusp) at the corresponding point.

Fig. 2 shows the result of the polar mapping qualitatively: The image of an inflection point $W_i$ of the curve $X$ is the cusp $w_i$ of the polar curve, the image of the double tangent $T$ is the double point $t$.

![Fig. 1. A polarity with respect to the complex unit circle assigns the point $P$ to the line $g$ with reciprocal radii.](image1)

![Fig. 2. Curve $X$ with inflection points and polar image with respect to the complex unit circle (origin $0$).](image2)
Now we will demonstrate an application of this theorem: First we consider a Bézier curve of order 4

\[ X(t) = \sum_{i=0}^{4} V_i B^4_i(t) \quad (t \in [0.1]) \]

with the Bézier points \( V_i \) and the Bernstein polynomials \( B^4_i(t) \) as basis functions. Further we choose the following special Bézier points with \( b \) as a parameter

\[ V_0(-12,0), \quad V_1(-6,6), \quad V(0,b), \quad V_3(6,3), \quad V_4(12,0). \]

The Bézier curve has a flat point if \( 2b_0 = 6 - \sqrt{24} \), i.e. the curve \( X \) has two inflection points for \( b < b_0 \), for \( b > b_0 \) \( X \) is convex. Fig. 3a contains these Bézier curves with the parameter \( b = b_0 \), \( b_0 \pm 1 \), \( b_0 \pm 2 \), Fig. 3b shows the corresponding polar curves. For \( b < b_0 \) it can be seen clearly that the polar curves have cusps. In order to magnify the curve region near the cusps, we shift the origin of our coordinate-system to \( \bar{0}(0; 1.5) \). By this translation nearby the expected inflection points we get an increase of the polar curves, so we can better recognize the effect of the polar mapping. Besides in Fig. 3b the scale was enlarged by the factor 10.

In order to demonstrate the elimination of undesirable inflection points by Theorem 1 we pick out the curve \( X \) with the parameter \( b = b_0 - 1 \). If we look at this curve alone it cannot be seen easily that \( X \) has inflection points while in the adjacent polar image the singularities can be observed clearly. Now we separate the Bézier curve \( X \) by the help of the Casteljau-algorithm into three segment curves. If we choose as boundary points of the segments the points of \( X \) with the parameter values \( t = 0.2 \) and \( t = 0.8 \), the middle segment contains the undesirable inflection points. Fig. 4a shows the new Bézier points. As coordinates of the Bézier points of the middle segment we get

\[ \bar{V}_0(-7.2; 2.54992), \quad \bar{V}_1(-3.36; 3.22901), \quad \bar{V}_2(0.48; 1.84539), \quad \bar{V}_3(4.32; 1.96895), \quad \bar{V}_4(16; 1.23152). \]

Now we choose as origin for the polarity the point \( \bar{0}(0; 1.5) \). In this mapping the middle segment we get

\[ \bar{V}_0(-7.2; 2.54992), \quad \bar{V}_1(-3.36; 3.22901), \quad \bar{V}_2(0.48; 1.84539), \quad \bar{V}_3(4.32; 1.96895), \quad \bar{V}_4(16; 1.23152). \]

![Fig. 3a. Family of Bézier curves of order 4 (\( b = b_0 + 2 \) corresponds to the top curve, \( b = b_0 - 2 \) corresponds to the bottom curve).](image)

![Fig. 3b. Polar mapping of the family of the Bézier curves in Fig. 3a.](image)
segment corresponds to the heavy curve in Fig. 4b; the two singularities can be seen clearly. To eliminate these singularities interactively we pick out the Bézier point $V_2$ and vary the ordinate to 2.2 – as Fig. 4b shows, the corresponding polar curve (the fine curve in Fig. 4b) has no singularities! Therefore undesirable inflection points on the Bézier curve have vanished (Fig. 4c).

A polarity maps the Bézier points $V_i$ onto the Bézier lines $g_i$. The lines $g_0$ and $g_4$ are tangents

![Fig. 4a. Segmentation of the second curve from bottom from Fig. 3a.](image1)

![Fig. 4b. Polar image of the curve in Fig. 4a (heavy) and the elimination of the singularities by changing $V_2$.](image2)

![Fig. 4c. Bézier curve with undesirable inflection points (heavy, see Fig. 4a) and the corrected Bézier curve without inflection points.](image3)

![Fig. 4d. Interpretation of the polar mapping of a Bézier curve as a dual Bézier curve and the corresponding Bézier lines.](image4)
Fig. 5a. Bézier curve of order 5 with one inflection point at least.

Fig. 5b. The polar image of the curve X in Fig. 5a shows that X has three inflection points.

Fig. 5c. The Bézier curve in Fig. 5a (heavy) is corrected to a Bézier curve without undesirable inflection points.

Fig. 5d. Polar image of the curves in Fig. 5c.
of the dual Bézier curve; the lines $g_1$ and $g_3$ as well as $g_2$ and $g_4$ intersect at the endpoints of the dual curve (see Fig. 4d).

As a further example we choose a Bézier curve of order 5 with 3 inflection points – two of them are undesirable because the given boundary tangents induce at least one inflection point. Fig. 5a contains our example: the Bézier curve $X$ and the Bézier points $V_i$

$$V_0(0; 0), \ V_1(3; 3), \ V_2(5; -2), \ V_3(7; 2) \ V_4(10; -2), \ V_5(12; 0).$$

If we choose as the origin of the polarity the point $O(7; -1)$, we get the polar image of $X$ as shown in Fig. 5b. The 3 singularities can be seen clearly. Further we can recognize that the polar curve $P$ has a pole because one tangent of $X$ passes through the origin. For eliminating the undesirable inflection points we again split up the Bézier curve $X$ by the help of Casteljau-algorithm into 3 segments. If we choose as boundary points of the segments the points of $X$ with the parameter $t = 0.1$ and $t = 0.9$, the middle segment contains the 3 inflection points. As coordinates of the corresponding Bézier points $V_i$ of the middle curve $X$ we get

$$V_0(1.409970; 0.853650), \ V_1(3.564327; 1.412955),$$

$$V_2(5.257445; -0.557005), \ V_3(7.004824; 0.709903),$$

$$V_4(9.192819; -0.858032), \ V_5(11.032613; -0.506274).$$

Fig. 5c contains (heavy) the segment $X$ and the corresponding Bézier points $V_i$, Fig. 5d contains the polar image $P$ of $X$ (heavy). Now we vary the ordinate of $V_3$ to 0.5 and the two undesirable inflection points vanish on the polar curve (see Fig. 5d, the fine curve). We can see the corrected curve $X$ including the corrected Bézier points in Fig. 5c (fine).

3. Smoothing of Bézier-surfaces

Now we will discuss surfaces and will point out how to recognize an undesirable zero of the Gaussian curvature or undesirable changes of sign and how these points or regions can be eliminated.

**Theorem 2.** If a surface with a regular parametric representation $X(u, v)$ has a zero or a change of sign of the Gaussian curvature at the point $(u_0, v_0)$ the polar image (polar surface) $P(u, v)$ has a singularity at the corresponding point.

**Remarks.** (1) [Bruce '81] has classified the singularities of the polar surface $P$. Singularities can be cusps, edges or dovetails.

(2) For the proof [Hoschek '83b] of this theorem the following representation of the polar surface $P$ can be used

$$P(u, v) = -\frac{[X_u, X_v]}{(X_u, X_v)} = \frac{(N, X_u, X_v)}{(X_u, X_v)}$$

with $N$ the normal vector of the surface $X$. At the singularities the outer product $[P_u, P_v]$ vanishes, which leads to $(N, N_u, N_v) = 0$. This determinant is exactly the numerator of a suitable representation of the Gaussian curvature [Laugwitz '68].

We will now apply this result to some examples: At first we will demonstrate the influence of the displacement of the origin upon the polar image: We consider a surface of revolution with the $z$-axis as axis of rotation. The meridian is a Bézier curve of 3rd order which lies in the $(x, z)$-plane and has the Bézier points

$$V_0(0; 0; 5), \ V_1(2; 0; 5), \ V_2(1; 0; 3), \ V_3(3; 0; 0).$$
Fig. 6a contains this surface in an oblique view, Figs. 6b, c show the corresponding polar surfaces with respect to the origin $\vec{0} = (0; 0; 0)$ and $\vec{0} = (0; 0; 3)$. The inflection point of the meridian has the $z$-coordinate $z \approx 3.26$ – in Figs. 6b, c it can be seen that the singularity can be recognized more detailed if the origin gets closer to the points on $X$ where the Gaussian curvature vanishes. Now we consider a tensor product Bézier surface

$$Z = \sum_{i=0}^{2} \sum_{k=0}^{4} V_{ik} B_i^2(u) B_k^4(v)$$

with the Bézier points

- $V_{00} (-2; -2; 0)$,
- $V_{01} (-1; -2; 2)$,
- $V_{02} (0; -2; -0.25)$,
- $V_{03} (1; -1; 2)$,
- $V_{04} (2; -2; 0)$,
- $V_{10} (-2; 0; 1)$,
- $V_{11} (-1; 0; 3)$,
- $V_{12} (0; 0; 0.75)$,
- $V_{13} (1; 0; 3)$,
- $V_{14} (2; 0; 1)$,
- $V_{20} (-2; 2; 0)$,
- $V_{21} (-1; 2; 2)$,
- $V_{22} (0; 2; -0.25)$,
- $V_{23} (1; 2; 2)$,
- $V_{24} (2; 2; 0)$.

Fig. 7a contains a tangential projection of this surface, Fig. 7b the corresponding polar image. In Fig. 7b it can easily be seen that $X$ is not convex. If we raise the $z$-component of $V_{02}$, $V_{12}$, $V_{22}$ by $\Delta z = 0.5$, the singularities vanish. Therefore the surface $X$ is now convex.
As another example we consider a tensor-product-Bézier surface of order 3 with the Bézier points

\[ \begin{align*}
V_{00}(0; 0; 0), & \quad V_{01}(1; 0; 1), & \quad V_{02}(2; 0; 2), & \quad V_{03}(3; 0; 0), \\
V_{10}(0; 1; 2), & \quad V_{11}(1; 1; 2), & \quad V_{12}(2; 1; 2), & \quad V_{13}(3; 1; 1), \\
V_{20}(0; 2; 1), & \quad V_{21}(1; 2; 2), & \quad V_{22}(2; 2; 2), & \quad V_{23}(3; 2; 2), \\
V_{30}(0; 3; 0), & \quad V_{31}(1; 3; 2), & \quad V_{32}(2; 3; 1), & \quad V_{33}(3; 3; 0).
\end{align*} \]

This surface has parameter curves which are convex without exception (see Fig. 8a). Nevertheless the surface is not convex! [Schelske '84] has given this surface while discussing convexity conditions of Bézier surfaces. The polar image (Fig. 8b) of \( X \) shows that the boundary regions have undesirable changes of sign of the Gaussian curvature. For smoothing such a region we will raise the \( z \)-component of \( V_{11} \) by \( \Delta z = 1 \), Fig. 8c then shows that the singularity in the neighbourhood of \( V_{11} \) has vanished. To eliminate the other singularities we can change the Bézier points \( V_{12}, V_{21}, V_{22} \) analogously.

4. Final remarks

We can extend our method without any difficulty to space curves [Hoschek '83b] or to other spline basis functions as B-spline functions [Boehm, Kahmann '83] or as Bezier polynomials over triangles [Farin '79].
Fig. 8a. Bézier surface $X$ with convex parameter curves.

Fig. 8b. Polar image of the surface $X$ in Fig. 8a.

Fig. 8c. Polar image with corrected lower region.

References


Coxeter, H.S.M. (1949), The real projective plane, New York.

Farin, G. (1979), Subsplines über Dreiecken, Dissertation Braunschweig.


